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EXISTENCE AND REGULARITY OF SOLUTIONS OF A NONLINEAR NONUNIFORMLY ELLIPTIC SYSTEM ARISING FROM A THERMISTOR PROBLEM

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Abstract A thermistor is an electric circuit device made of ceramic material whose electric conductivity depends on the temperature. If the only heat source is the electric heating, the temperature and the electric potential satisfy a nonlinear elliptic system which is also degenerate if the electric conductivity is not uniformly bounded from above or away from zero. Under general boundary conditions, we establish existence and Hölder continuity of solutions of such a nonlinear nonuniformly elliptic system. When the electric conductivity linearly depends on the temperature, we provide a non-uniqueness and non-existence example.

Key Words Thermistor; elliptic system; nonlinear; nonuniformly elliptic; mixed boundary value problem.

Classification 35J70, 35J55.

1. Introduction

A thermistor, or a thermally-sensitive-resistor, is an electrical device made of semiconducting materials whose electrical resistivity changes up to 5 orders of magnitude as the temperature increases over a certain range. It has many applications such as current regulation, switching, thermal conductivity analysis, and control and alarm; see, for example, Hyde [12] and Llewellyn [13].

When acting as a (renewable) circuit breaker, a thermistor operates as follows: an increase in current provides more (electrical) heating, leading to a rise in temperature of the material which causes a rise in the resistivity, thereby reducing the current (to almost zero if the temperature increases beyond a critical limit). When the thermistor cools down, its resistivity decreases and the normal operation of the circuit resumes. In this paper, however, we shall only study the steady state problems.

Denote by Ω the domain in R^N occupied by the thermistor ($N = 2, 3$ are cases of physical interest) and by $u, \phi, \sigma(u)$, and $k(u)$ the temperature, electrical potential, electrical conductivity, and thermal conductivity, respectively. Then the steady state thermistor problem is to solve the elliptic system

$$\nabla(\sigma(u)\nabla\varphi)=0 \quad \text{in } \Omega \quad \text{(conservation of current)} \tag{1.1}$$

$$-\nabla(k(u)\nabla u)=\sigma(u)|\nabla\varphi|^2 \quad \text{in } \Omega \quad \text{(conservation of energy)} \tag{1.2}$$

subject to the boundary conditions

$$\varphi=\varphi_0 \quad \text{on } \Gamma_D^\varphi, \quad \partial_n\varphi=0 \quad \text{on } \Gamma_N^\varphi\equiv\partial\Omega\setminus\overline{\Gamma_D^\varphi} \tag{1.3}$$

$$u=u_0 \quad \text{on } \Gamma_D^u, \quad \partial_n u+h(x,u)=0 \quad \text{on } \Gamma_N^u\equiv\partial\Omega\setminus\overline{\Gamma_D^u} \tag{1.4}$$

where ∂_n is the outward normal derivative and $\Gamma_D^\varphi, \Gamma_D^u$ are smooth hypersurfaces. Typically, Γ_D^φ consists of two disjoint hypersurfaces Γ_D^1 and Γ_D^2, and

$$\varphi=V \quad \text{on } \Gamma_D^1, \quad \varphi=0 \quad \text{on } \Gamma_D^2$$

where V is the voltage difference applied on the thermistor.

There has been recent mathematical interest in this thermistor problem in both the case when σ is positive [4, 5, 6, 11, 15, and the references therein] and the case when σ is vanishes at large temperature [1, 2, 3, 10, 16].

The obstacles in this thermistor problem are the quadratic growth on the right-hand side of (1.2) and the degeneracy of (1.1) when $\sigma(u)$ is not uniformly bounded from above or away from zero.

In this paper we shall consider the case when $\sigma(u)$ is positive but is not necessarily uniformly bounded from above and away from zero as $u\to\infty$. Since the change of thermal conductivity is of secondary importance, we shall assume that $k=1$. In fact, the method given here can be applied to the general case of $k>0$ as well. We shall establish the existence and Hölder continuity of the solution of (1.1)–(1.4) under certain conditions on $\sigma(u)$ and $h(x,u)$.

In the case when $k=1$ and σ is uniformly bounded from above and away from zero, existence of weak solutions to (1.1)–(1.4) was recently established by Howison, Rodrigues, and Shillor [11]. The strategy they used to get around the quadratic growth is to write $\sigma|\nabla\varphi|^2$ as $\nabla(\sigma(\varphi-\varphi_0)\nabla\varphi)+\sigma\nabla\varphi_0\nabla\varphi$ which is a bounded functional on $H^1(\Omega)$. They proved the boundedness of the solution only for the case of Dirichlet boundary condition or for the case of $N=2$ where one can apply Meyers' theorem on the elliptic equations of type (1.1) to deduce that $\nabla\varphi\in L^p(\Omega)$ for some $p>2$, and therefore to deduce that $u\in W^{2,p/2}\subset C^{2-4/p}$ by the L^p estimate and the Sobolev imbedding theorem. They also established the uniqueness of solutions for the case when the solutions are sufficient "small". The general uniqueness problem, however, is still open.

When $\sigma(u)/k(u)$ is not uniformly bounded away from zero, equations (1.1), (1.2) with Dirichlet boundary data was studied in [4, 5, 6, 15]. The strategy here is to use