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EXISTENCE OF A WEAK SOLUTION FOR THE PHASE CHANGE PROBLEM WITH JOULE'S HEATING

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Abstract A phase change problem with Joule's heating describes the processes of electric heating in a conducting material. It is modeled as a coupled system of nonlinear partial differential equations with quadratic growth in the gradient. We establish the existence of a weak solution for the problem in two dimensions.

Key Words phase change; system of nonlinear partial differential equations; quadratic growth.
Classification 35K

1. Introduction

In this paper we consider a model that describes the combined effects of heat and electrical current flows in a metal. When an electrical current flows across the matal, Joule heating is generated by the resistance of the metal to the electrical current, which brings about the increase of the temperature. A phase change will take place once the melting temperature is crossed and the latent heat is absorbed.

Let $u = u(x,t)$ denote the temperature, u_* the melting temperature, $h = h(x,t)$ be the enthalpy density, $\varphi = \varphi(x,t)$ the electrical potential and $\sigma = \sigma(u)$ be the temperature dependent electrical conductivity. The mathematical model for the evolution under consideration is the following nonlinear system:

Find a triplet $\{h, u, \varphi\}$ such that

$$\frac{\partial h}{\partial t} - \Delta u = \sigma(u)|\nabla\varphi|^2 \tag{1.1}$$

$$\nabla(\sigma(u)\nabla\varphi) = 0 \tag{1.2}$$

$$h \subset u + \lambda H(u - u_*) \tag{1.3}$$

and the initial and boundary conditions, where

$$H(s) = \begin{cases} -1 & \text{if } s < 0 \\ [-1,1] & \text{if } s = 0 \\ 1 & \text{if } s > 0 \end{cases} \tag{1.4}$$

When $h \equiv u (i.e. \lambda = 0)$ in (1.1)–(1.3), Cimatti [1] proved the existence of weak solutions in two space dimensions and Chipot and Cimatti [2] proved the uniqueness for the problem in one and two space dimensions. For the physical background and the known results for the problem (1.1)–(1.3) we refer to [3] for more details and the references therein. In [3] by using regularization and time discretization the existence of the solutions $\{u_n, \varphi_n\}$ for the discretized approximated problems is proved, and then the strong convergence of $\{u_n\}$ and $\{\varphi_n\}$ in L^2 is proved. But we find that the proof of the latter step includes a mistake and the method breaks down. Here we shall give a new proof of the existence for the problem in two space dimensions.

The plan of the paper is as follows. In Section 2 the definition of the weak solution and the main result are stated. In Section 3 an approximating problem is solved by using Schauder fixed-point theorem. Further a priori estimates on the approximating solutions are obtained is Section 4. Since the right term of (1.1) involves the quadratic growth in the gradient of φ, we will use Meyers' estimate [4] to obtain the higher integratility of $|\nabla\varphi|$ and then prove the local equicontinuity of $\{u_n\}$ by using the modified method of the De Giorgi estimates (see [5]). In Section 5 it will be concluded that there exists a sequence of approximating solutions converging to the weak solution of the problem under consideration.

2. The Definition of the Weak Solutions and the Main Result

Let Ω be a smooth bounded domain of $\mathcal{R}^2$, which is occupied by a conducting material. Denote $\Omega_T = \Omega \times (0,T)$. We shall adopt the notation and symbol in [7] and make the following assumptions.

$$\sigma(s) \in C^1(\mathcal{R}^1),\ 0 < \sigma_* \le \sigma(s) \le \sigma^* < +\infty\ \forall s \in \mathcal{R}^1 \tag{2.1}$$

$$u_0(x) \in C(\bar{\Omega}),\ u_0(x) = 0 \text{ on } \partial\Omega,\ u_0(x) \neq u_* \text{ a.e. in } \Omega,\ u_* > 0 \tag{2.2}$$

$$\varphi_0 \in C^{1+\alpha,0}(\bar{\Omega}_T)\ (0 < \alpha < 1) \tag{2.3}$$

The enthalpy formulation of the problem is as follows:

Problem (P): Determine a triplet $\{h, u, \varphi\}$ such that

$$h \in \alpha(u) \quad \text{in } \Omega_T \tag{2.4}$$

$$\frac{\partial h}{\partial t} - \Delta u = \sigma(u)|\nabla\varphi|^2 \quad \text{in } \Omega_T \tag{2.5}$$

$$u = 0 \quad \text{on } \partial\Omega \times [0,T] \tag{2.6}$$

$$u = u_0(x) \quad \text{on } \Omega \times \{0\} \tag{2.7}$$

$$\nabla(\sigma(u)\nabla\varphi) = 0 \quad \text{in } \Omega_T \tag{2.8}$$

$$\varphi = \varphi_0 \quad \text{on } \partial\Omega \times [0,T] \tag{2.9}$$