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GLOBAL HÖLDER CONTINUOUS SOLUTIONS OF NONSTRICTLY HYPERBOLIC SYSTEMS

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Abstract This paper considers the Cauchy problem for nonstrictly hyperbolic systems $u_t + \frac{1}{2}(au^2 + v^2)_x = 0$, $v_t + (uv)_x = 0$ and gives the Hölder continuous solutions under some stronger restrictions of data by applying the method of vanishing viscosity, where $a > 2$ is a constant.

Key Words Nonstrictly hyperbolic systems; Hölder continuous solutions; vanishing viscosity.

Classification 35L.

1. Introduction

The general 2×2 systems of conservation laws with quadratic flux functions

$$\begin{cases} u_t + \frac{1}{2}(a_1u^2 + 2b_1uv + c_1v^2)_x = 0 \\ v_t + \frac{1}{2}(a_2u^2 + 2b_2uv + c_2v^2)_x = 0 \end{cases} \tag{1.1}$$

are of interest because their solutions may approximate the solutions of general 2×2 systems of conservation laws

$$\underset{\sim}{u}_t + \underset{\sim}{f}(\underset{\sim}{u})_x = 0 \tag{1.2}$$

where $\underset{\sim}{u} = (u, v)$, $\underset{\sim}{f} = (f, g)$, in a neighborhood of an isolated hyperbolic singularity. Such a singularity is an isolated point. In its neighborhood (1.2) are strictly hyperbolic and at the point the Jacobian

$$A(\underset{\sim}{u}) = \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix}$$

has equal eigenvalues and is diagonalizable. It is shown by Shearer and Schaeffer [1] that when systems (1.1) are hyperbolic, there is a nonsingular linear change of dependent variables which transforms systems (1.1) into

$$\begin{cases} u_t + \frac{1}{2}(au^2 + 2buv + v^2)_x = 0 \\ v_t + \frac{1}{2}(bu^2 + 2uv)_x = 0 \end{cases} \tag{1.3}$$

The systems (1.3) with $b = 0$ are called symmetric since the solutions have so called both up-down symmetry and left-right symmetry [2]. The Riemann problem has been solved entirely in papers [2-5] and the interaction of shock waves is given in paper [6]. Here it is interesting to study more general initial values which guarantee the existence of global Hölder continuous solutions.

Let F be the mapping from E^2 into E^2 defined by

$$F : (u, v) \to \left(\frac{1}{2}(au^2 + v^2), uv\right)$$

and denote by $dF(u, v)$ the Frechet derivative (Jacobian) of F. Two eigenvalues of dF are

$$\lambda_1 = \frac{1}{2}(a+1)u - S, \quad \lambda_2 = \frac{1}{2}(a+1)u + S \tag{1.4}$$

with corresponding right and left eigenvectors

$$\begin{aligned} r_1 &= \left(S - \frac{1}{2}(a-1)u, -v\right)^T, \quad r_2 = \left(S + \frac{1}{2}(a-1)u, v\right)^T \\ l_1 &= \left(S - \frac{1}{2}(a-1)u, -v\right), \quad l_2 = \left(S + \frac{1}{2}(a-1)u, v\right) \end{aligned} \tag{1.5}$$

where

$$S = \sqrt{\left(\frac{1}{2}(a-1)u\right)^2 + v^2}$$

We have by simple calculation

$$d\lambda_1(r_1) = S - \frac{1}{2}(a-1)u + \frac{a-1}{2S}\left(S - \frac{1}{2}(a-1)u\right)^2 + \frac{v^2}{S} \tag{1.6}$$

$$d\lambda_2(r_2) = S + \frac{1}{2}(a-1)u + \frac{a-1}{2S}\left(S + \frac{1}{2}(a-1)u\right)^2 + \frac{v^2}{S} \tag{1.7}$$

Therefore it follows from (1.4) that $\lambda_1 = \lambda_2$ at point $(0, 0)$ and the strict hyperbolicity fails to hold. That the first characteristic field is linearly degenerate for $u \geq 0$, $v = 0$ and the second degenerate for $u \leq 0$, $v = 0$ follows from (1.6) and (1.7).

There have been many papers [7, 8] about the study of the global smooth solutions to strictly hyperbolic systems of two conservation laws. However it is more difficult and important to study nonstrictly hyperbolic systems. In this article we study the Hölder continuous solutions of the Cauchy problem for the symmetric system

$$\begin{cases} u_t + \frac{1}{2}(au^2 + v^2)_x = 0 \\ v_t + (uv)_x = 0 \end{cases} \tag{1.8}$$

by means of a new method which is a variant of the "viscosity" argument. In the standard "viscosity" argument one tries to construct solutions of (1.8) as limits ($\varepsilon \to$