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UNIQUENESS OF THE SOLUTIONS OF $u_t = \Delta u^m$ AND $u_t = \Delta u^m - u^p$ WITH INITIAL DATUM A MEASURES: THE FAST DIFFUSION CASE

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Abstract In this paper, we study the Cauchy problems

$$u_t = \Delta u^m \quad u(x,0) = \mu$$

and

$$u_t = \Delta u^m - u^p \quad u(x,0) = \mu$$

where $p > 0$, $m > \left(1 - \frac{\alpha}{n}\right)^+$ and μ is a finite Radon measure. We prove the uniqueness of solution and the existence of solution.
Key Words Porous medium equation; Cauchy problem; initial datum a measure; uniqueness.
Classification 35K.

1. Introduction

In this paper, we consider the Cauchy problems

$$u_t = \Delta u^m \qquad \text{in } S_T = R^n \times (0,T) \tag{1.1}$$

$$u(x,0) = \mu \qquad \text{in } R^n \tag{1.2}$$

and

$$u_t = \Delta u^m - u^p \qquad \text{in } S_T \tag{1.3}$$

$$u(x,0) = \mu \qquad \text{in } R^n \tag{1.4}$$

where $n \geq 1$, $m > \left(1 - \frac{2}{n}\right)^+$, $p > 0$, $\mu \in M^+(R^n)$; $M(R^n)$ (resp. $M^+(R^n)$) is the set of finite (resp. and nonnegative) Radon measures on R^n.

Equation (1.1) arises in many applications. We will not recall them here, since they can be found in many papers, for example [1] [2]. It is also a model of physical phenomena when the initial datum is a measure (see [3]).

For the cases of regular diffusion ($m = 1$) and slow diffusion ($m > 1$) it has been shown in respectively [4], [5] that the problem (1.3) (1.4) has a unique solution when either $m = 1$, $p > 0$, or $m > 1$, $p \geq 1$.

The object of this paper is to extend these results to the case when

$$m > \left(1 - \frac{2}{n}\right)^+, \quad p > 0$$

where $(S)^+ = \max(0, S)$. We will prove

Theorem 1 *Let* $\left(1 - \frac{2}{n}\right)^+ < m < 1$. *Then Cauchy problem* (1.1)-(1.2) *has a unique solution.*

Theorem 2 *Let* $m > \left(1 - \frac{2}{n}\right)^+$, $0 < p < m + \frac{2}{n}$.

Then problem (1.3) (1.4) *has a unique solution.*

Clearly Theorem 2 generalizes the result in [5]. For simplicity, we prove only Theorem 2 for

$$\left(1 - \frac{2}{n}\right)^+ < m < 1$$

In fact, when $m > 1$, the proof of Theorem 2 needs only a minor change.

Note that for $m < 1$, $u \in L^1(\boldsymbol{R}^n) \cap L^\infty(\boldsymbol{R}^n)$ and $u \geq 0$ do not imply $u^m \in L^1(\boldsymbol{R}^n)$ and $u^{m-1} \in L^\infty(\boldsymbol{R}^n)$ in general, and u^p is not Lipschitz continuous for $0 < p < 1$. Thus there are some differences between the proofs of Theorems 1, 2 and that in [5], [6].

2. Proof of Theorem 1

Definition 2.1 *A solution u of* (1.1)(1.2) *is a nonnegative function defined in S_T such that*

(a) $u \in L^1(S_T) \cap L^\infty(\boldsymbol{R}^n \times (s,T)) \quad \forall s \in (0,T)$,

(b) $u_t = \Delta u^m$ *in* $D(S_T)$,

where in $D'(S_T)$ means in the sense of distributions in S_T

(c) *for every* $x \in C_0^\infty(\boldsymbol{R}^n)$

$$\operatorname*{limess}_{t \to 0+} \int_{\boldsymbol{R}^n} u(t)\chi = \int_{\boldsymbol{R}^n} \chi\mu \tag{2.1}$$

We denote by $C_0(\boldsymbol{R}^n)$ (resp. $C_b(\boldsymbol{R}^n)$) the set of continuous function on $\boldsymbol{R}^n$ with compact support (resp. bounded).

Definition 2.2 *A sequence $\mu_n \in M^+(\boldsymbol{R}^n)$ is said to be converging to μ in* $\sigma(M(\boldsymbol{R}^n), C_0(\boldsymbol{R}^n))$ (resp. $\sigma(M(\boldsymbol{R}^n), C_b(\boldsymbol{R}^n))$ *if for any* $\phi \in C_0(\boldsymbol{R}^n)$ (resp. $\phi \in C_b(\boldsymbol{R}^n)$)

$$\lim_{n \to \infty} \int_{R^n} \phi\mu_n = \int_{R^n} \phi\mu \tag{2.2}$$

Definition 2.3 *For $u, f \in D'(\boldsymbol{R}^n)$, $\alpha > 0$ we say $u = K_a * f$ if*

$$\alpha u - \Delta u = f \quad \text{in } D'(\boldsymbol{R}^n) \tag{2.3}$$