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HYPOELLIPTICITY OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract In this paper, we study the hypoellipticity problems for fully nonlinear partial differential equations of order m. For a solution $u \in C^\rho_{\text{loc}}(\Omega)$, if the linearized operator for the nonlinear equation on u satisfies some subelliptic conditions, we can deduce $u \in C^\infty(\Omega)$ by using the paradifferential operator theory of J. -M. Bony.
Key Words Hypoellipticity; paradifferential operators; para-linearization; parametrix.
Classification 35H05.

0. Introduction

This paper deals with the C^∞ regularity of the solution of the following mth order nonlinear partial differential equation:

$$F[u] = F(x, u(x), \cdots, \partial^\beta u(x), \cdots)_{|\beta| \le m} = 0 \tag{0.1}$$

where $x \in \Omega$, Ω is an open set of R^n, F is a real C^∞ function.

If $u \in C^\rho_{\text{loc}}(\Omega)(\rho > m)$ is a real solution for the equation (0.1), we define an associate linearized operator:

$$P(x, D) = \sum_{|\alpha| > 2m-\rho} a_\alpha(x) \partial^\alpha_x \tag{0.2}$$

where $a_\alpha(x) = \dfrac{\partial}{\partial u_\alpha} F(x, u(x), \cdots, \partial^\beta u(x), \cdots)$ are all real functions in $C^{\rho-m}_{\text{loc}}(\Omega)$.

In [1], Xu Chaojiang discussed the case $m = 2$. Suppose that the second order linearized operator satisfies the Oleinik-Radkevic conditions, he proved the hypoellipticity of nonlinear equation. Similar to the linear equation, he obtained C^∞ regularity of the solution by constructing *a priori* estimates for the linearized operator.

In this paper, we study the hypoellipticity of (0.1) and obtain the following main theorem.

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Theorem 0.1 *Suppose that $u \in C^{\rho}_{\text{loc}}(\Omega)$ is a real solution of Equation* (0.1), $0 \le m' \le m$, $0 \le \delta < \frac{1}{2}$ *and* $\rho > m + 1 + \frac{1}{1-2\delta}(m - m')$, *and the symbol of the linearized operator defined by* (0.2) *satisfies the following conditions:*

H_1) *For every compact subset $K \subset \Omega$, there exist constants $R > 0$, $C_1 > 0$, $C_2 > 0$, such that*

$$C_1|\xi|^{m'} \le |p(x,\xi)| \le C_2|\xi|^m \quad \textit{for all } x \in \boldsymbol{K}, \xi \in \boldsymbol{R}, |\xi| \ge R$$

H_2) *For every compact subset $K \subset \Omega$, $\alpha, \beta \in N^n$, $|\beta| < \rho - m$, there exist constants $R > 0$, $C_{\alpha,\beta,K} > 0$, such that*

$$|\partial_x^\beta \partial_\xi^\alpha p(x,\xi)| \le C_{\alpha,\beta,K}|p(x,\xi)|\ |\xi|^{-|\alpha|+\delta|\beta|} \quad \textit{for all } x \in \boldsymbol{K}, \xi \in \boldsymbol{R}^n,\ |\xi| \ge R$$

Then the solution $u \in C^\infty(\Omega)$.

Here conditions H_1) and H_2) are similar to those in the case of linear operator (see [2]), therefore this paper will use similar method as well, i.e., to construct the parametrices. However, the critical problem here is that the coefficients of the linearized operator $P(x, D)$ are not C^∞. Because of this, we shall use the paradifferential operator of J. -M. Bony [3] to prove our result. First, we make the linearization of Equation (0.1) so that it becomes a linear paradifferential equation. Secondly, we construct the relevant symbol spaces and operator classes, which will help us to prove the existence of "left parametrix" of the paradifferential operator, thus we obtain the regularity of the nonlinear equations.

The plan of this paper is as follows: In Section 1, we study the nonhomogeneous symbolic calculus and construct the corresponding symbol spaces and operator classes, which will help us to obtain the "left parametrix" in this frame. Proof of the main results will be given in Section 2. Finally, an example of nonlinear hypoellipticity will be discussed in Section 3.

1. Nonhomogeneous Symbolic Calculus

Let us first recall the following well-known results:

(1) Suppose that $a(x) \in L^\infty(\boldsymbol{R}^n)$, supp $\hat{a}(\xi) \subset \{|\xi| \le \lambda\}$, then $a(x) \in C^\infty(\boldsymbol{R}^n)$, and for any $\alpha \in N^n$, there exists a constant $C(n, \alpha)$, such that

$$\|\partial_x^\alpha a\|_{L^\infty(\boldsymbol{R}^n)} \le C(n,\alpha)\lambda^{|\alpha|}\|a\|_{L^\infty(\boldsymbol{R}^n)}$$

(2) for any constants $0 < \varepsilon_1 < \varepsilon_2 < 1$, $R > 0$, there exists a function $\psi(\eta, \xi) \in C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^n)$, such that $\psi = 0$ if $|\eta| \ge \varepsilon_2|\xi|$, $\psi = 1$ if $|\eta| \le \varepsilon_1|\xi|$ and $|\xi| \ge R$, and for any α, $\beta \in N^n$, there exists a constant $C_{\alpha,\beta} > 0$, such that

$$|\partial_\eta^\alpha \partial_\xi^\beta \psi(\eta,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{-|\alpha|-|\beta|} \tag{1.1}$$

Actually, we can choose any cut-off function in the definition of the paradifferential operator which satisfies the condition (1.1).