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THE CAUCHY PROBLEM OF THE POROUS MEDIUM EQUATION WITH ABSORPTION*

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Abstract In this paper we study the existence of solution for the following Cauchy problem

$$\begin{cases} u_t = \Delta u^m - u^p \\ u(x,0) = u_0(x) \end{cases}$$

We show how the growth condition of initial trace is determined by the absorption.

Key Words Existence of solution; Cauchy problem; porous media equation.
Classification 35K.

1. Introduction

In this paper we consider the Cauchy problem

$$u_t = \Delta u^m - u^p \quad \text{in } S_T = R^N \times (0,T) \tag{1.1}$$

$$u(x,0) = \mu \tag{1.2}$$

where $m \geq 1$, $p > 1$ and μ is a nonnegative σ-finite Borel measure in R^N.

Equation (1.1) arises from many applications. We will not recall them here, since they can be found in many papers, for example in [1] [2]. The case when the initial datum is a measure is also a model of physical phenomena (see [3] [4]). In this paper we are interested in some features of initial traces for the solutions of the porous medium equations with absorption. It is well known (see [4]) that for the porous medium equation

$$u_t = \Delta u^m \tag{1.3}$$

the initial trace μ of a nonnegative solution must satisfy

$$\int_{B_R} d\mu \leq C R^{N+\frac{2}{m-1}}, \quad m > 1, \ R > 1 \tag{1.4}$$

$$\int_{B_R} \exp\left\{-\frac{|x|^2}{4T}\right\} d\mu < \infty, \quad m = 1 \tag{1.5}$$

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and any nonnegative solution of (1.3) satisfies a Harnack inequality

$$\int_{B_R} u(x,0)dx \le \gamma\Big\{R^{N+\frac{2}{m-1}}T^{-\frac{1}{m-1}} + T^{\frac{N}{2}}(u(0,T))^{\frac{N}{2}[m-1]+1}\Big\} \tag{1.6}$$

where the constant γ does not depend on u, R, T and

$$B_R = \{x \in \mathbf{R}^N : |x| < R\}$$

In this paper, we will show the differences between (1.1) and (1.3) and how the growth condition of initial trace for the solution of (1.1) is determined by the absorption.

Definition 1.1 *A function u is said to be a weak solution of (1.1)–(1.2) on S_T if u satisfies*

1. $u \in C(S_T) \cap L^\infty(\mathbf{R}^N \times (\tau, T))$ *for every* $\tau > 0$;
2. $\iint_{S_T} (u\eta_t + u^m \Delta\eta - u^p\eta)dxdt = 0$ *for all* $\eta \in C_0^2(S_T)$;
3. $\lim_{t\to 0^+} \int_{\mathbf{R}^N} \phi(x)u(x,t)dx = \int_{\mathbf{R}^N} \phi(x)d\mu$ *for all* $\phi \in C_0(\mathbf{R}^N)$.

We will prove the following theorems.

Theorem 1 *Suppose that $m < p < m + \frac{2}{N}$ and let μ be a nonnegative σ-finite Borel measure on $\mathbf{R}^N$. Then problem (1.1)–(1.2) has a solution.*

Theorem 2 *Let $m < p < m + \frac{2}{N}$. Then problem (1.1)–(1.2) has a unique solution u with $u_t \in L^1_{\mathrm{loc}}(S_T)$.*

Theorem 3 *Suppose that $1 < m < p$ and $u \in L^\infty_{\mathrm{loc}}(\mathbf{R}^N)$ with*

$$\varliminf_{|x|\to\infty} \frac{\mu(x)}{|x|^\alpha} > 0$$

where $\alpha > \frac{2}{m-p}$. Then

$$t^{\frac{1}{p-1}}u(x,t) \to C^* \text{ as } t \to \infty$$

uniformly on sets of the form:

$$\{x \in \mathbf{R}^N : |x| < \alpha t^{\frac{1}{\beta}}\}$$

in which

$$C^* = \Big(\frac{1}{p-1}\Big)^{\frac{1}{p-1}}, \quad \beta = \frac{2(p-1)}{p-m}$$

Theorem 1 shows that if $p > m$, problem (1.1)–(1.2) has a global solution in time for any nonnegative σ-finite Borel measure μ on $\mathbf{R}^N$. This is in sharp contrast with the case (1.3). Moreover, by Lemma 2.3 in Section 2 of this paper the solution of (1.1)–(1.2) has a bound on $\mathbf{R}^N \times (\tau, T), \forall\tau > 0$, which does not depend on the initial trace. Thus the solutions of (1.1) do not satisfy Harnack type inequality similar to (1.6). In