J.Partial Diff. Eqs., 7 (1994), 257–263
©International Academic Publishers Vol.7, No.3

EXISTENCE OF HARMONIC FUNCTIONS WITH FINITE ENERGY ON COMPLETE RIEMANNIAN MANIFOLDS*

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(Received Feb. 1, 1992; revised June 30, 1992)

Abstract Let M be a noncompact complete Riemannian manifold. We consider the existence of harmonic functions with $|\nabla u| \in L^p(M)$.

Key Words Harmonic function; finite energy; noncompact manifold.
Classification 58E20.

1. Introduction

Let M be a noncompact complete Riemannian manifold of dimension n. Yau[1] proved that every nonnegative $L^p (1 < p < \infty)$ subharmonic function on M is constant. Later, Li and Schoen[2] proved that if the Ricci curvature of M is nonnegative, then every nonnegative L^p $(0 < p \leq 1)$ subharmonic function on M is also constant (see also[3]). Dodziuk[4] considered the relations between the geometry or topology of a manifold and the spaces of L^2 harmonic forms on it.

In this paper, we consider the existence of harmonic functions with $|\nabla u| \in L^p(M)$. If M has nonnegative Ricci curvature, we know[5] that $\Delta|\nabla u|^2 \geq 0$ when u is a harmonic function, so there is no nonconstant harmonic function with $|\nabla u| \in L^p(M)$ $(0 < p < \infty)$ by Yau's result and the result of Li-Schoen (Example 2 in [4] implies that there is not any nonconstant harmonic function u with $|\nabla u| \in L^2(M)$ on $M = R \times S^1$ with the product metric). For this reason, we mainly consider the case where M is a negatively curved manifold. We first give an example similar to Example 1 in [4] to show that there exists a complete Riemannian manifold M with sectional curvature K_M satisfying $-1 \leq K_M \leq 0$, on which there does exist a nonconstant harmonic function u with $|\nabla u| \in L^p(M)$ for any $1 < p \leq \infty$. For every $0 < p < 1$, we then give a complete Riemannian manifold M with Ricci curvature $\mathrm{Ric}(M)$ satisfying $\mathrm{Ric}(M) \sim -Cr(x)^{-2}$ as $r(x) \to \infty$, where $r(x)$ is the geodesic distance from x to some fixed point in M

* The research supported by National Natural Science Foundation of China.

and $C > 2$, on which there is a nonconstant harmonic function u with $|\nabla u| \in L^p(M)$. Finally we prove the following theorem.

Theorem 1 *Let M be a simply connected complete Riemannian manifold of dimension n with constant negative curvature $-K^2$.*

(1) *If u is a harmonic function and $|\nabla u| \in L^p(M)$ for $1 \le p \le n-1$, then u =const.*

(2) *There exists a nonconstant harmonic function u with $|\nabla u| \in L^p(M)$ for any $n-1 < p < \infty$.*

For the nonexistence result we have a more general theorem (Theorem 2 in Section 3). For the existence result we suspect that if the sectional curvature K_M satisfies $-K_2^2 \le K_M \le -K_1^2 < 0$ we may have a result similar to the second part of Theorem 1, but we think new ideas are needed to prove it.

2. Two Examples

Example 1 Suppose $D^2 \subset R^2$ is the unit disc, $ds_0^2 = dx_1^2 + dx_2^2$, $ds^2 = \dfrac{4}{(1-x_1^2-x_2^2)^2} ds_0^2$, $M_0 = (D^2, ds_0^2)$, $M_1 = (D^2, ds^2)$. Let T^{n-2} be the $n-2$ torus with a flat metric. We set $M = M_1 \times T^{n-2}$ with the product metric.

Clearly $K_{M_1} \equiv -1$, so, $-1 \le K_M \le 0$. $\Delta_M = \Delta_{M_1} + \Delta_{T^{n-2}}$.

We choose $u(x_1, x_2, \cdots, x_n) = x_1$, then $\Delta_{M_0} u = 0$ and $\Delta_{M_1} u = 0$.

$$\int_M |\nabla_M u|^p dV_M = \int_{T^{n-2}} \int_{M_1} |\nabla_{M_1} u|^p dV_{M_1} dV_{T^{n-2}}$$

$$= \int_{T^{n-2}} \int_{M_0} |\nabla_{M_0} u|^p \varphi^{2-p} dV_{M_0} dV_{T^{n-2}}$$

where $\varphi(x) = \dfrac{2}{1-x_1^2-x_2^2}$, $|\nabla_{M_0} u| = 1$.

We therefore have $|\nabla_M u| \in L^p(M)$ for any $1 < p \le \infty$.

Example 2 ([2]) Let M_0 be a compact surface with arbitrary genus. Assume the metric on M_0 around some point $O \in M_0$ is flat. Hence locally around O we can write the metric in polar coordinates as $ds_0^2 = dt^2 + t^2 d\theta^2$. Consider the Green's function on M_0 with the pole at O, $G(0, x) = f(x)$. By definition $f(x)$ is harmonic on $M \backslash \{O\}$ with respect to the given metric ds_0^2. Let $ds^2 = \rho^2 ds_0^2$ ($\rho > 0$) be a conformally changed metric on M_0, and let $M_1 = (M_0, ds^2)$. Obviously, $\Delta_{M_1} f(x) = 0$.

Choose $\rho(t) = t^{-1}\Big(\log \dfrac{1}{t}\Big)^{-\alpha}$. $\dfrac{1}{2} < \alpha < 1$, we know [2] that M_1 is a complete Riemannian manifold with sectional curvature $K(x)$ satisfying $K(x) \sim -\alpha[(1-\alpha)r(x)]^{-2}$ as $r(x) \to \infty$ where $r(x)$ is the geodesic distance from x to some fixed point in M_1.

Let $M = M_1 \times T^{n-2}$. We also have $\Delta_M f(x) = 0$. Since [6] $|\nabla_{M_0} f(x)| \le \dfrac{C}{t}$,