J.Partial Diff. Eqs., 7 (1994), 289–322
©International Academic Publishers Vol.7, No.4

PARABOLIC Q-MINIMA AND THEIR APPLICATION*

Zhou Shulin
(Department of Mathematics, Peking University, Beijing 100871, China)
(Received Nov. 2, 1992; revised May 6, 1993)

Abstract In this paper, the notion named parabolic Q-minima is endowed with rich meanings and its local behavior is investigated. As its direct application we obtain the local regularity, such as boundedness, continuity, Hölder continuity of the weak solutions of the various filtration equations, e.g., the equation of Newtonian polytropic filtration, the general equation of Newtonian filtration, the equation of elastic filtration, the equation of non-Newtonian polytropic filtration. Therefore, a unifying approach to various regularity results for a great number of nonlinear degenerate parabolic equations is provided.

Key Words Parabolic Q-minima; boundedness; continuity; Hölder continuity.
Classifications 35B45, 35K55, 35K65.

1. Introduction

1.1 The definition of parabolic Q-minima and examples

The notion of Q-minima in the elliptic case was defined and studied by M. Giaquinta and E. Giusti [1], while the parabolic counterpart was introduced by W. Wieser [2] and extended by the author [3]. In this paper the notion of parabolic Q-minima is generalized further such that it includes the weak solutions or their certain functions, of the equations of both Newtonian filtration and non-Newtonian filtration. Therefore, our results about parabolic Q-minima naturally hold for the weak solutions mentioned above.

First of all we state some standard notation, which will be quoted later. Denote Ω a bounded domain in R^N, $\Omega_T = \Omega \times (0,T)$ $(T > 0)$, $z = (x,t)$ any point in Ω_T. C (sometimes $\tilde{C}, \bar{c}, c, c_1, c_2$) are generic constants which may change from line to line. $W^{1,p}$, $\overset{\circ}{W}{}^{1,p}$, $\mathcal{D}, \mathcal{D}'$ are the usual spaces.

And now we define the notion of parabolic Q-minima.

Definition An n-dimensional vector function $v \in L^p_{\rm loc}(0,T;W^{1,p}_{\rm loc}(\Omega)) \cap L^\gamma_{\rm loc}(\Omega_T)$ is called a parabolic Q-minimum if there exist a constant $Q \geq 1$, a measurable vector function $\Phi : R^n \to R^n$ such that $\Phi(v) \in L^1_{\rm loc}(\Omega_T)$, and a Carathéodory vector function

* The project supported in part by National Natural Science Foundation of China and in part by Natural Science Foundation of Peking University.

$F = F(z, w, q) : \Omega \times (0, T) \times R^n \times R^{nN} \to R^n$, such that for every $\phi \in C_0^\infty(\Omega_T)$, the following inequality

$$-\int_K \Phi(v)\frac{\partial \phi}{\partial t}dz + E(v; K) \leq QE(v - \phi; K) \tag{1.1}$$

holds, here $K = spt\phi$, $dz = dxdt$, and

$$E(v; K) = \int_K F(z, v, \nabla v)dz \tag{1.2}$$

where F satisfies the growth condition:

$$\lambda_1|q|^p - b|w|^\gamma - g(z) \leq F(z, w, q) \leq \lambda_2|q|^p + b|w|^\gamma + g(z) \tag{1.3}$$

where $\lambda_1, \lambda_2 > 0$, $b \geq 0$, $p > 1$, $\gamma > 0$, and $g(z)$ is a non-negative integrable vector function.

To extend the applicability of the notion, we call v a B-restricted parabolic Q-minimum if (1.1) holds merely for all $\phi \in B$, a certain subset contained in $C_0^\infty(\Omega_T)$. For $Q = 1$ in (1.1) and $F = F(x, w, q)$, we call these Q-minima minimal solutions ([2]).

Remark 1.1 If v is a parabolic Q-minimum with Q, Φ, F, then $-v$ is a parabolic Q-minimum with $Q, \tilde{\Phi}, \tilde{F}$ where $\tilde{F}(z, w, q) = F(z, -w, -q)$, $\tilde{\Phi}(s) = -\Phi(-s)$.

Remark 1.2 The notion of our parabolic Q-minima is wider than those in [2] and [3], which are respectively the special cases when we set $p = 2$, $\Phi(s) = s$ and $\Phi(s) = s$.

We assert that the weak solutions or their certain functions (in adequate Banach spaces), of a large class of equations are parabolic Q-minima when Q, Φ, F are chosen properly.

Example 1 $u_t = \Delta(|u|^{m-1}u)$ $(m > 1)$ ([4]).

Here $v = |u|^{m-1}u$ is a parabolic Q-minimum with $\Phi(s) = |s|^{\frac{1}{m}(1-m)}s$, $F(z, w, q) = \frac{1}{2}|q|^2$, $Q = 1$.

Example 2 $u_t = \Delta\varphi(u)$ ([4]).

Here $v = \varphi(u)$ is a parabolic Q-minimum with $\Phi(s) = \varphi^{-1} \circ (s)$($\varphi^{-1}$ is the inverse function), $F(z, w, q) = \frac{1}{2}|q|^2$, $Q = 1$.

Example 3 $u_t = \nabla \cdot (|\nabla u|^{k-1}\nabla u)$ $(k > 0)$ ([4]).

Here $v = u$ is a parabolic Q-minimum with $\Phi(s) = s$, $F(z, w, q) = \frac{1}{1+k}|q|^{k+1}$, $Q = 1$.

Example 4 $u_t = \nabla \cdot (|\nabla(|u|^{m-1}u)|^{k-1}\nabla(|u|^{m-1}u))$, $(m > 1, k > 0)$ ([4]).

Here $v = |u|^{m-1}u$ is a parabolic Q-minimum with $\Phi(s) = |s|^{\frac{1}{m}(1-m)}s$, $F(z, w, q) = \frac{1}{k+1}|q|^{k+1}$, $Q = 1$.

Example 5 $u_t = \nabla \cdot (|\nabla\varphi(u)|^{k-1}\nabla\varphi(u))$

Here $v = \varphi(u)$ is a parabolic Q-minimum with $\Phi(s) = \varphi^{-1} \circ (s)$ (φ^{-1} is the inverse function), $F(z, w, q) = \frac{1}{k+1}|q|^{k+1}$, $Q = 1$.