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EXISTENCE UNIQUENESS AND DECAY OF THE GLOBAL WEAK SOLUTIONS FOR A CLASS OF PARABOLIC SYSTEMS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract In this paper, a class of parabolic systems with nonlinear boundary conditions is discussed. By introducing a complete metric space with decay of W_p^s-norm, we obtain the existence uniqueness of global weak solutions, our method is simpler than before. A decay estimate of the global weak solutions is obtained simultaneously.

Key Words Nonlinear boundary conditions; global weak solutions; decay.
Classification 35K.

1. Introduction

In this paper, we deal with the global existence, uniqueness and decay of the weak solution for the following problem:

$$\begin{cases} u_t - D\Delta u + Cu = f(u) & \text{in } \Omega \times (0, +\infty) \\ D\dfrac{\partial u}{\partial \nu} + b(x)u = g(u) & \text{on } \partial\Omega \times (0, +\infty) \\ u(0, x) = u_0(x) & \text{on } \Omega \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in R^n, $u = (u^1, u^2, \cdots, u^N)$ is an N-vector valued function, $D = \text{diag}\ (d_1, d_2, \cdots, d_N)$, $(d_i > 0, i = 1, 2, \cdots, N)$, $C = \text{diag}\ (c_1, c_2, \cdots, c_N)$, $b(x) = \text{diag}\ (b_1(x), b_2(x), \cdots, b_N(x))$ are $N \times N$ matrices, $\dfrac{\partial u}{\partial \nu}$ is the outward normal derivative on Ω.

Recently, Amann [1] has obtained an important generalized variation-of-constants" formula for the following problem:

$$\begin{cases} u_t + \mathcal{A}(t)u = f(t, u) \\ B(t)u = g(t, u) \\ u(x, 0) = u_0(x) \end{cases} \tag{1.2}$$

which is a general problem of (1.1). Using this formula, and based on the existence of local weak solutions and *a priori* bound of the local weak solutions. Amann [1] has obtained a global existence result for the weak solutions of (1.2).

Other authors have also discussed some reaction-diffusion systems with nonlinear boundary conditions. Under weaker nonlinearity (see [2]) or quasimonotone conditions (see [3]–[5]) of f, g, some existence results of the solutions have been obtained.

In this paper, when $C_i (i = 1, 2, \cdots, N)$ in (1.1) is a suitable positive number and the initial value is small, by introducing a complete metric space with the decay of W_p^s-norm and using "generalized variation-of-constants" formula, we prove the global existence, uniqueness of the weak solutions of (1.1) under some assumptions of f, g which are different from [1], [2]. Our method is simpler than before. Simultaneously, we obtain a decay estimate of the global weak solution which has not been discussed in [1], [2].

This paper is organized as follows, in Section 2, we collect the important results of [1]. In Section 3, we prove the main results of this paper and give two examples to explain that all assumptions on f, g and C are neccessary to the decay of global weak solutions. In Section 4, we apply our results to an enzyme system.

2. "Generalized Variation-of-Constants" Formula —Integral Equation Equivalent to (1.1)

In this section, we use the notations in [1] and collect the important results of [1]. Throughout this paper, all vector spaces are over complex field $\mathbb{C}^1$, we usually do not distinguish between Banach spaces differing only by equivalent norms. Let X, Y be Banach spaces, we write $X \overset{d}{\hookrightarrow} Y$ if X is continuously and densely embedded in Y. We denote by $\mathcal{L}(X, Y)$ the vector space of continuous linear operators from X to Y and we denote by Isom (X, Y) the set of all isomorphisms in $\mathcal{L}(X, Y)$. We write $A \in \xi(X, M, -\omega)$ if the linear operator $-A$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{-tA} | t \geq 0\}$ on X such that $\|e^{-tA}\| \leq Me^{-\omega t}, t \geq 0$. If $-A$ generates a strongly continuous analytic semigroup on X, then we write $A \in \mathcal{H}(X)$.

Let Ω be a bounded domain in $\boldsymbol{R}^n$ of class C^2, that is, $\bar{\Omega}$ is an n-dimensional C^2-submanifold of $\boldsymbol{R}^n$ with boundary $\partial\Omega$.

For $1 < p < +\infty$, we let

$$L_p := L_p(\Omega, \mathbb{C}^N) = [L_p(\Omega, \mathbb{C}^1)]^N$$

and put

$$\left.\begin{aligned} W_p^s &:= W_p^s(\Omega, \mathbb{C}^N) = [W_p^s(\Omega, \mathbb{C}^1)]^N \\ W_p^s(\partial\Omega) &:= W_p^s(\partial\Omega, \mathbb{C}^N) = [W_p^s(\partial\Omega, \mathbb{C}^1)]^N \end{aligned}\right\} s \in \boldsymbol{R}, \quad 1 < p < +\infty$$

where $\mathbb{C}^N$ is N-dimensional complex space, $W_p^s(\Omega, \mathbb{C}^1)$ and $W_p^s(\partial\Omega, \mathbb{C}^1)$ are the standard Sobolev spaces.