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CONDENSATION OF LEAST-ENERGY SOLUTIONS OF A SEMILINEAR NEUMANN PROBLEM*

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Abstract This paper is devoted to the study of the least-energy solutions of a singularly perturbed Neumann problem involving critical Sobolev exponents. The condensation rate is given when $n > 4$ and an asymptotic behavior result is obtained.

Key Words Neumann problem; least-energy solutions.
Classification 35B.

1. Introduction

This paper is devoted to the study of the condensation behavior of the least-energy solutions, as $d \to 0$, of the following singularly perturbed semilinear Neumann problem

$$\begin{cases} d\Delta u - u + u^{\tau} = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial v} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Delta = \sum_{j=1}^{n} \dfrac{\partial^2}{\partial x_j^2}$ is the Laplace operator, Ω is a bounded smooth domain in R^n, $n \geq 3$, v is the unit outer normal to $\partial\Omega$, $\tau = \dfrac{n+2}{n-2}$ and $d > 0$ is a constant. By a *least-energy solution* of (1.1) we mean a (classical) solution of (1.1) which minimizes the "energy" functional

$$J_d(u) = \int_\Omega \left\{ \frac{1}{2}(d|\nabla u|^2 + u^2) - \frac{1}{\tau+1} u_+^{\tau+1} \right\} dx$$

where $u_+ = \max(u, 0)$, among all the solutions of (1.1). Such problems have been studied by many authors, see, e.g., [1], [2] and references therein.

It was proved in [3] that the least-energy solution u_d of (1.1) must exhibit "singular point-condensation" character on the boundary $\partial\Omega$ as $d \to 0$. That is, $u_d \to 0$ in Ω

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as $d \to 0$, the (global) maximum of u_d in $\overline{\Omega}$ is assumed exactly at one point P_d which must lie on the boundary $\partial\Omega$ and $\|u_d\|_{L^\infty(\Omega)} \to \infty$ as $d \to 0$.

The purpose of this paper is to establish the condensation rate and the location of the condensation points of u_d as $d \to 0$, and give a detailed description of the convergence under various scalings, in the case when $n > 4$. Throughout this paper, u_d will always denote a least-energy solution of (1.1), α_d and P_d will always denote the maximum and the maximum point of u_d in $\overline{\Omega}$, respectively, i.e. $u_d(P_d) = \|u_d\|_{L^\infty(\Omega)} = \alpha_d$. Let $\beta_d = \alpha_d^{-\frac{2}{n-2}}$.

Before stating our main results, we recall Theorem 3.1, in [3] as follows. Let

$$U(x) = \left[1 + \frac{|x|^2}{n(n-2)}\right]^{-\frac{n-2}{2}}, \quad x \in R^n \tag{1.2}$$

which is a solution of

$$\Delta U + U^\tau = 0 \tag{1.3}$$

in R^n satisfying $U(0) = 1$. Let

$$S = n(n-2)\pi\left[\Gamma\left(\frac{n}{2}\right)\Big/\Gamma(n)\right]^{2/n} \tag{1.4}$$

which is the best Sobolev constant in R^n in the following sense:

$$S = \inf\left\{\int_\Omega |\nabla u|^2 dx : u \in H_0^1(\Omega) \text{ and } \int_\Omega |U|^{\tau+1} dx = 1\right\} \tag{1.5}$$

Denote $B_\delta(P) = \{x \in R^n : |x - P| < \delta\}$.

Theorem A [3] *Let u_d be a least-energy solution of (1.1). Then for d sufficiently small the maximum of u_d in $\overline{\Omega}$ is attained exactly at one point P_d which must lie on the boundary $\partial\Omega$, and we have*

(i) $\|u_d\|_{L^\infty(\Omega)} \to \infty$ *as* $d \to 0$;

(ii) $u_d \to 0$ *everywhere in* Ω *as* $d \to 0$;

(iii) $d^{-\frac{n}{2}} \int_\Omega u_d^{\tau+1} dx \to \frac{1}{2} S^{n/2}$ *as* $d \to 0$.

Furthermore, for any $\varepsilon > 0$ there exist two positive constants $d_0 = d_0(\Omega, \varepsilon)$ and $R = R(\Omega, \varepsilon)$ such that for $0 < d < d_0$ the following estimates hold:

(iv) $\left|\frac{u_d(x)}{\|u_d\|_{L^\infty(\Omega)}} - U\left[\frac{\Psi_d(x)}{\beta_d\sqrt{d}}\right]\right| < \varepsilon$ *for all* $x \in \Omega \cap B_{\beta_d\sqrt{d}R}(P_d)$;

(v) $u_d(x) < C\varepsilon \exp\left(-\gamma_0\zeta(x)/\sqrt{d}\right)$ *for all* $x \in Q\backslash B_{\sqrt{d}R}(P_d)$.

where U is given by (1.2), Ψ_d is a diffeomorphism straightening a boundary portion of $\partial\Omega$ around P_d (as described in Section 2), $\zeta(x) = \min\left\{\eta_0, \operatorname{dist}\left(x, \partial\Omega \cap B_{\sqrt{d}R}(P_d)\right)\right\}$, and C, γ_0, η_0 are positive constants only depending on Ω.

Remark 1.1 From the proof of Lemma 3.35 in [3] we actually see that, for *any* $\delta > 0$ and any $\varepsilon > 0$ there is a $d_0 > 0$ such that for $0 < d < d_0$ the estimate (v) holds in $\Omega\backslash B_{\sqrt{d}\delta}(P_d)$.