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EXISTENCE OF BOUNDED SOLUTIONS FOR QUASILINEAR SUBELLIPTIC DIRICHLET PROBLEMS*

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Abstract This paper proves the existence of solution for the following quasilinear subelliptic Dirichlet problem:

$$\begin{cases} \sum_{j=1}^{m} X_j^* a_j(x, v, Xv) + a_0(x, v, Xv) + H(x, v, Xv) = 0 \\ v \in M_0^{1,p}(\Omega) \cap L^\infty(\Omega) \end{cases}$$

Here $X = \{X_1, \cdots, X_m\}$ is a system of vector fields defined in an open domain M of R^n, $n \geq 2$, $\Omega \subset\subset M$, and X satisfies the so-called Hörmander's condition at the order of $r > 1$ on M. $M_0^{1,p}(\Omega)$ is the weighted Sobolev's space associated with the system X. The Hamiltonian H grows at most like $|Xv|^p$.

Key Words Subelliptic equation; weighted Sobolev's space; Dirichlet problem.
Classification 35H.

1. Introduction

This paper is concerned with the existence of bounded solutions for the following quasilinear boundary value problem:

$$\begin{cases} A(v) + H(x, v, Xv) = 0 & (1.1) \\ v \in M_0^{1,p}(\Omega) \cap L^\infty(\Omega) & (1.2) \end{cases}$$

where $X = \{X_1, \cdots, X_m\}$ is a system of real smooth vector fields defined in an open domain M of R^n, $n \geq 2$, $\Omega \subset\subset M$. The principal part A is a differential operator of second order in divergence form of Leray-Lions:

$$A(v) = \sum_{j=1}^{m} X_j^* a_j(x, v, Xv) + a_0(x, v, Xv) \tag{1.3}$$

The Hamiltonian H grows at most like $|Xv|^p$.

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We assume that the system $X = \{X_1, \cdots, X_m\}$ satisfies the following Hörmander's condition:

$$\text{(H)} \quad \begin{cases} X \text{ together with their commutators } X_\alpha = [X_{\alpha_1}, \cdots [X_{\alpha_{s-1}}, X_{\alpha_s} \cdots]] \\ \text{up to some fixed length } r \text{ span the tangent space at each point of } M. \end{cases}$$

$M_0^{1,p}(\Omega)$ is the associate weighted Sobolev's space (see Section 2).

We assume that a_j, a_0, H, are Carathéodory functions, which mean

$$\forall (s,\xi) \in \boldsymbol{R} \times \boldsymbol{R}^n, \ x \to a_j(x,s,\xi) \text{ are measurable} \tag{1.4}$$

$$a.e. \ x \in \Omega, \ (s,\xi) \to a_j(x,s,\xi) \text{ are continuous} \tag{1.5}$$

And they satisfy the following conditions for all $s \in \boldsymbol{R}, \xi, \xi^* \in \boldsymbol{R}^n$, and for almost every $x \in \Omega$,

$$|a_j(x,s,\xi)| \leq \beta[k(x) + |s|^{p-1} + |\xi|^{p-1}], \quad j = 0, 1, \cdots, m \tag{1.6}$$

$$|H(x,s,\xi)| \leq c_0 + b(|s|)|\xi|^p \tag{1.7}$$

$$\sum_{j=1}^m [a_j(x,s,\xi) - a_j(x,s,\xi^*)](\xi_j - \xi_j^*) > 0, \quad \xi \neq \xi^* \tag{1.8}$$

$$\sum_{j=1}^m a_j(x,s,\xi)\xi_j \geq \alpha|\xi|^p \tag{1.9}$$

$$a_0(x,s,\xi)s \geq \alpha_0|s|^p \tag{1.10}$$

where $\alpha, \alpha_0, \beta, c_0$ are strictly positive constants, $k(x)$ is a given positive function in $L^{p'}(\Omega)$, $1 < p < +\infty$, $1/p + 1/p' = 1$, and $b(s)$ is a positive, increasing, continuous function defined on $\boldsymbol{R}^+$. Since the system of vector fields $X = \{X_1, \cdots, X_m\}$ is degenerate of order $r > 1$, the equation (1.1) is called subelliptic.

We shall prove the following existence and regularity theorem.

Theorem 1 *Assume that $\partial\Omega$ is C^∞, and non characteristic for the system of vector fields X. Then, under the hypotheses* (H) *and* (1.4)–(1.10), *there exists at least one solution of problem* (1.1) *and* (1.2).

As in Theorem 7 of [1], we can also prove that this weak solution is in the class $C^\tau_{\text{loc}}(\Omega)$ for some $\tau > 0$.

The proof of Theorem 1 consists in the following steps. We first define an approximate problem. We then prove *a priori* estimates in $L^\infty(\Omega)$ and $M_0^{1,p}(\Omega)$ for the solutions of these approximate problems. Finally we prove that the sequence of solutions of approximate problems is compact in the strong topology of $M_0^{1,p}(\Omega)$, a result which allows to pass to the limit and to obtain the existence result.

The existence result does not require any smoothness assumptions on the coefficients a_j, $j = 0, 1, \cdots, m, H$. We could also consider other types of boundary conditions.