J.Partial Diff. Eqs., 8(1995), 108–114
©International Academic Publishers Vol.8, No.2

A NOTE ON A CLASS OF STRONGLY COUPLED SEMILINEAR REACTION-DIFFUSION EQUATIONS

Tian Dagang
(Dept. of Math. Phys., Wuhan University of Technology, Wuhan 430070, China)
(Received Oct. 10, 1992; revised Jan. 24, 1994)

Abstract A class of strongly coupled semilinear reaction-diffusion equations is discussed. The FitzHugh-Nagumo equation in [1] is only an example. The results that strongly coupled term generates an analytic semigroup extended the results in [4]. The results of the existence and uniqueness of solution and the stability of zero solution improved that in [1]. This paper also discusses the existence of periodic solution.
Key Words Strongly coupled; semilinear; semigroup.
Classification 35K.

1. Introduction

Because there is strongly coupled term and nonlinear term, the FitzHugh-Nagumo equations (see [1]–[3]) are always a challenging and noticeable problem. In this paper, a class of strongly coupled semilinear reaction-diffusion equations to which the FitzHugh equations are only special cases is discussed. We discuss when the strongly coupled operator generates an analytic semigroup that extends the results in [4]. For the existence and uniqueness of the solution and the stability of zero solution of the equations, our results improve the results in [1]. Moreover, we also discuss the existence of periodic solution.

Let X be a Banach space. For a linear operator $A : D(A) \to X$, we denote by $D(A)$ the domain of A, by $\rho(A)$ the resolved set of A, by $\sigma(A)$ the spectrum of A respectively.

For a real $n \times n$ matrix $Q, Q = (q_{ij})_{n\times n}$, we define a strongly coupled linear operator $QA : D(QA) \to X^n$ by $QA = (q_{ij}A)_{n\times n}$.

Let A be a sector operator (see [5]), $S_{a,\phi}$ be the relevant sector.

Lemma 1 *Suppose that the characteristic values of Q are real and the algebraic multiplicity and the geometric multiplicity of zero characteristic value are equal, then*

$$\rho(QA) = \bigcap_{i=1}^{m} \lambda_i \rho(A) - \{0\}$$

where $\{\lambda_j\}_1^m$ are the non-zero characteristic value of Q.

Proof Let $Q = PJP^{-1}$, where $J = \mathrm{diag}\{J_1, \cdots, J_k, 0\} = \mathrm{diag}\{\overline{J}, 0\}$ is the Jordan type of Q. Evidently, $QA = PJP^{-1}A = PJAP^{-1}$, $\rho(QA) = \rho(JA)$. We need only to show that

$$\rho(\overline{J}A) = \bigcap_{i=1}^{m} \lambda_i \rho(A)$$

we have $\overline{J}A - \lambda I = \mathrm{diag}\{J_1A - \lambda I, \cdots, J_kA - \lambda I\}$ and when $\lambda \in \bigcap_{i=1}^{m} \lambda_i\rho(A)$, there is $\mu_i \in \rho(A)$, such that $\lambda = \lambda_i\mu_i$, $i = 1, 2, \cdots, m$. Now that the $\lambda_i(A - \mu_i I)$ is invertible, we have

$$(J_iA - \lambda I)^{-1} = \begin{bmatrix} \lambda_i^{-1}(A-\mu_iI)^{-1} & -\lambda_i^{-2}(A-\mu_iI)^{-2}A & \cdots & (-1)^{n_i-1}\lambda_i^{-n_i}(A-\mu_iI)^{-n_i}A^{n_i-1} \\ 0 & \lambda_i^{-1}(A-\mu_iI)^{-1} & \cdots & (-1)^{n_i-2}\lambda_i^{-n_i+1}(A-\mu_iI)^{-n_i+1}A^{n_i-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_i^{-1}(A-\mu_iI)^{-1} \end{bmatrix}_{n_i\times n_i}.$$

since $(A - \mu_i I)^{-1}A = I + \mu_i(A - \mu_i I)^{-1}$ and A is a sector operator, $(JA - \lambda I)^{-1} = \mathrm{diag}\{(J_iA - \lambda I)^{-1}, \cdots, (J_kA - \lambda I)^{-1}\}$ is bounded. Hence

$$\bigcap_{i=1}^{m} \lambda_i\rho(A) \subset \rho(\overline{J}A)$$

Suppose that $\lambda \in \rho(\overline{J}A)$. Consider the right-hand lower element of $(J_iA - \lambda I)^{-1}$, we know $\lambda/\lambda_i \in \rho(A)$, $i = 1, 2, \cdots, k$, hence $\lambda \in \bigcap_{i=1}^{m} \lambda_i\rho(A)$.

Theorem 1 *Suppose that Q satisfies the conditions of Lemma 1, and its characteristic values are nonnegative. Then QA is a sector operator. Hence it can generate an analytic semigroup.*

Proof Let $a_1 < \min\{0, a\}$, where a is the vertex of sector $S_{a,\phi}$. Let $b = \max\limits_{1\le i\le m} \lambda_i a_1$, then by Lemma 1, when $\lambda \in S_{b,\phi} \subset \rho(QA)$, $\lambda/\lambda_j \in S_{a_1,\phi}$, $j = 1, 2, \cdots, m$, and $\|(\lambda_iA - \lambda I)^{-1}A\|$ are bounded. Since

$$\|(QA - \lambda I)^{-1}\| \le C\sqrt{\sum_{i=1}^{k}\sum_{j=0}^{n_i-1}(n_i - j)\|(\lambda_iA - \lambda I)^{-j-1}A^j\|^2 + |\lambda|^{-(n-m)}}$$

$$\le C\sqrt{\sum_{i=1}^{k}|\lambda_i|^{-2}\left\|\left(A - \frac{\lambda}{\lambda_i}\right)^{-1}\right\|^2\sum_{j=0}^{n_i-1}(n_i - j)\|(\lambda_iA - \lambda I)^{-j}A^j\|^2 + |\lambda|^{-(n-m)}}$$

there are constants $M > 0$, such that, for every $\lambda \in S_{b,\phi}$

$$\|(QA - \lambda I)^{-1}\| \le \frac{M}{|\lambda - b|}$$

Hence QA is a sector operator.

Remark Lemma 1 in [4] is a special case of Theorem 1.