J.Partial Diff. Eqs., 8(1995), 126–134
©International Academic Publishers Vol.8, No.2

DECAY OF SOLUTION OF A PARABOLIC EQUATION IN 2-SPACE DIMENSIONS

Guo Boling and Zhang Linghai
(Institute of Applied Physics and Computational Mathematics, P.O. Box: 8009, Beijing, China)

(Received May 14, 1993; revised Sept. 3, 1993)

Abstract We present a simple method for verifying the uniform L^1 bound and establish sharp rates of L^2 decay of the global solution to the initial value problem for a 2-dimensional parabolic equation.

Key Words Decay estimates; 2D parabolic equation.
Classification 35Q20.

1. Introduction

Newton-Boussinesq equations are of the form

$$\Delta\Psi_t + J(\Delta\Psi, \Psi) = \Delta^2\Psi - \alpha\theta_x \tag{1}$$

$$\theta_t + J(\theta, \Psi) = \beta\Delta\theta \tag{2}$$

which describe the famous Benard flow. In these last equations, Ψ is flow function, $\Delta\Psi$ is vortex, $\left(\frac{\partial\Psi}{\partial y}, -\frac{\partial\Psi}{\partial x}\right)$ is velocity vector, θ is the temperature, $J(\theta, \Psi) = \frac{\partial\theta}{\partial x}\frac{\partial\Psi}{\partial y} - \frac{\partial\theta}{\partial y}\frac{\partial\Psi}{\partial x}$, $\alpha > 0, \beta > 0$ are constants.

The global existence of the generalized and classical solutions have been established. Spectral method and nonlinear Galerkin methods for solving two-dimensional Newton-Boussinesq equations have been discussed [1–2]. Universal attractors for the Benard problem, existence and physical bounds on their fractal dimension have also been investigated [3].

We note that if we put $\Phi = \Delta\Psi$, and neglect the effect of the temperature in (1), we get

$$\Phi_t + J(\Phi, \Psi) = \Delta\Phi \tag{3}$$

We are interested in the long time behavior of the global solution to the initial value problem for the following parabolic equation

$$\theta_t + J(\theta, \Psi) = \beta\Delta\theta,\ \beta > 0 \tag{4}$$

$$\theta(x,y,0)=\theta_0(x,y) \tag{5}$$

where $\theta(x,y,t), \Psi(x,y,t)$ are known scalar functions of the real variables $-\infty < x$, $y<\infty$, $0\le t<\infty$; $\beta>0$ is a constant.

In this paper, we want to establish sharp L^2 decay of the global solution to problem (1–2), with initial data $\theta_0(x)\in L^1\cap L^2$. The decay results follow from the *a priori* L^1, L^2 integral estimates and the Fourier transform. The standard argument relies on a technique that involves the splitting of the phase space into two time-dependent domains. For this information, please refer to [4–7].

There has been considerable literature on decay of solutions to the initial value problems for nonlinear evolution equations. Schonbek studied decay of solutions to parabolic conservation laws

$$U_t+\sum_{k=1}^{n}\frac{\partial}{\partial x_k}f_k(U)=\varepsilon\Delta U,\quad U(x,0)=U_0(x) \tag{6}$$

She established that if $U_0(x)\in L^1\cap H^2$, then

$$\int_{R^n}|U|^2dx\le C(1+t)^{-n/2}$$

Moreover, under the assumption

$$\left|\frac{d}{dU}f_k(U)\right|\le C|U|^p,\ p\ge 1+\frac{4}{n},\ |U|\le 1$$

or

$$|f_k(U)|\le C(U)^q,\ q\ge 2\left(1+\frac{1}{n}\right),\ |U|\le 1$$

she established the L^∞ optimal decay estimate

$$\|U(t)\|_\infty\le C(1+t)^{-n/2}$$

Readers who are interested in this problem can find other similar works on solutions to nonlinear evolution equations in our references.

Denote by $Q_t=\{(x,y,s):-\infty<x,\ y<\infty,\ 0\le s\le t\}$, where $0\le t<\infty$.

For simplicity, we will denote by C any positive constant which depends only on the norms of the initial data θ_0, and the positive constant β, but never depends on $t\ge 0$. Moreover, we regard

$$\|U(t)\|=\|U(t)\|_{L^2(R^2)},\quad \|U(t)\|_\infty=\|U(t)\|_{L^\infty(R^2)}$$
$$\|U(t)\|_m=\|U(t)\|_{H^m(R^2)},\quad \|U_0\|=\|U_0\|_{L^2(R^2)},\quad \|U_0\|_m=\|U_0\|_{H^m(R^2)}$$

Suppose that $f(x,y)\in L^1(R^2)\cap L^2(R^2)$, define its Fourier transform as follows

$$F[f](\zeta,\eta)=\hat{f}(\zeta,\eta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\exp[-ix\zeta-iy\eta]dxdy$$

As usual, the definition is extended by continuity to the space of tempered distributions.