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CAUCHY PROBLEM FOR SEMILINEAR WAVE EQUATIONS IN FOUR SPACE DIMENSIONS WITH SMALL INITIAL DATA*

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Abstract In this paper, we consider the Cauchy problem

$$\Box u(t,x) = |u(t,x)|^p, \quad (t,x) \in R^+ \times R^4$$
$$t = 0 : u = \varphi(x), \quad u_t = \psi(x), \quad x \in R^4$$

where $\Box = \partial_t^2 - \sum_{i=1}^{4} \partial_{x_i}^2$ is the wave operator, $\varphi, \psi \in C_0^\infty(R^4)$. We prove that for $p > 2$ the problem has a global solution provided the initial data is sufficiently small.
Key Words Wave equation; Cauchy problem; global solution.
Classification 35L.

1. Introduction

In this paper, we consider the Cauchy problem

$$\Box u(t,x) = |u(t,x)|^p, \quad (t,x) \in R^+ \times R^n \tag{1.1}$$
$$t = 0 : u = \varphi(x), \quad u_t = \psi(x), \quad x \in R^n \tag{1.2}$$

where $\Box = \partial_t^2 - \sum_{i=1}^{n} \partial_{x_i}^2$ is the wave operator, $\varphi, \psi \in C_0^\infty(R^n)$.

The question under consideration is for what values of p do sufficiently small data leads to a global solution. In three space dimensions, John [1] has obtained the decisive result that for $p > 1 + \sqrt{2}$ all data that are sufficiently small in a certain norm lead to global existence and for $1 < p < 1 + \sqrt{2}$ the solution tends to infinity in finite time for any nontrivial initial data of compact support. Moreover, it is conjectured that the critical value $P_0(n)$, generalizing John's result to n space dimensions, should be the positive root of

$$(n-1)\chi^2 - (n+1)\chi - 2 = 0$$

One half of this conjecture has been verified, Glassey [2] and Sideris [3] respectively in 2 and $n \geq 4$ space dimensions showed that the global solution of (1.1) (1.2) does not

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exist if $p < P_0(n)$ under some positivety condition on the initial data. Still, Glassey [4] showed in two space dimensions that the global small solutions of (1.1) (1.2) exist if $p > P_0(2)$. However, the question of global existence of small solutions of (1.1) (1.2) when $p > P_0(n)$ and $n \geq 4$ is still open. It seems that Sideris has solved the problem for the radial data, but the paper is unavailable to us. The problem seems to be difficult for the general data, see Y. Choquet-Bruhat [5]. In this paper, we shall solve the problem in four space dimensions. The main technical difficulty with the higher dimensional problem lies in the fact that the fundamental solution of the wave operator is no longer positive, consequently, the weighted L^∞ estimate as has been used by John [1] can not carry through. In this paper, instead of weighted L^∞ estimate, we use a weighted Sobolev L^2 norm based on the idea of Klainerman [6].

It should be mentioned that in the critical case $p = P_0(n)$, Schaeffer [7] has proved that the solution of (1.1) (1.2) blows up in finite time provided $n = 2, 3$ and

$$\int \psi(x)dx > 0$$

Actually, any nontrivial initial data leads to finite time blow-up of solutions for $p = P_0(3)$ in three space dimensions, see Zhou [8]. Beyond three space dimensions, the question is still open.

Finally, we remark that our problem is related to the small data scattering problem for (1.1). In fact, the parallel method as applied in this paper can be used to prove the existence of the wave operator in case $p > P_0(4)$, see Pecher [9].

2. Preliminaries

Following Klainerman [6], we introduce a set of partial differential operators

$$\Gamma = (D, L, \Omega)$$

where D is defined as

$$D = (\partial_t, \partial_{x_1}, \cdots, \partial_{x_n}) \tag{2.1}$$

and

$$L = (L_a, a = 0, 1, \cdots, n) \tag{2.2}$$

with

$$L_0 = t\partial_t + x_1\partial_{x_1} + \cdots + x_n\partial_{x_n} \tag{2.3}$$

$$L_i = t\partial_{x_i} + x_i\partial_t, \quad i = 1, \cdots, n \tag{2.4}$$

and

$$\Omega = (\Omega_{ij}, \ i, j = 1, \cdots, n) \tag{2.5}$$

with

$$\Omega_{ij} = x_i\partial_{x_j} - x_j\partial_{x_i}, \quad i, j = 1, \cdots, n \tag{2.6}$$