J.Partial Diff. Eqs. 8(1995), 145–158
©International Academic Publishers
Vol.8, No.2

THE REGULARITY FOR SOLUTIONS OF VARIATIONAL INEQUALITIES*

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(Received Mar. 2, 1992; revised June 6, 1994)

Abstract The regularity for solutions of elliptic equations is rather perfectly solved. But it does not so perfect for that of elliptic variational inequalities. In literature only different special situations are considered. Now the boundedness, $C^{0,\lambda}$ continuity and $C^{1,\alpha}$ regularity are proved for solutions of one-sided obstacle problems under more general structural conditions, in which the growth orders of u are permitted to reach the critical exponents and the growth order γ of the gradient in B is permitted to be super critical as $1 < p < n$.

Key Words Elliptic variational inequality; one-sided obstacle problem; boundedness; Hölder continuity; $C^{1,\alpha}$ regularity.

Classification 35J85.

Let G be a bounded domain in n-dimensional Euclidean space E^n. Consider on G the following obstacle problem:

$$\int_G \{(\nabla v - \nabla u) \cdot A(x, u, \nabla u) + (v - u)B(x, u, \nabla u)\}dx \geq 0, \ \forall v \in K(\psi) \qquad (0.1)$$

where $K(\psi) = \{u | u \in \overset{\circ}{W}{}^{1,p}(G) + u_0 \subset W^{1,p}(G)$ and $u(x) \geq \psi(x)$ for a.e. x in $G\}$. For the case of $p > 1$, $A(x, u, \xi) = |\xi|^{p-2}\xi$ and $|B(x, u, \xi)| \leq C(|\xi|^{p-1} + |u|^{p-1} + 1)$ Choe[1] has proved the $C^{1,\alpha}$ regularity for the solution of (0.1). The same results have been obtained by Lindqvist[2] and Fuchs[3] for $p \geq 2$ and $B = 0$ and by Norando[4] for $p \geq 2$ and other one-sided obstacle problems. In Michael-Ziemer[5] the $C^{0,\lambda}$ continuity of the solution is proved for A and B satisfying the structural conditions (1.1)–(1.3) with $q = p$ and $\gamma = p - 1$. Based on [5] Liang-Yu[6] and Wang-Liang[7] have proved the $C^{1,\alpha}$ regularity for $p > 2$ and for $1 < p < 2$, respectively. The structural conditions on A and B appearing in both [6] and [7] are the same as in [5]. In this paper we consider more general structural conditions: the exponent q appearing in (1.1)–(1.3) is permitted to reach the critical Sobolev exponent p^* and the growth order γ of ξ in B is permitted to be super critical, that is $p - 1 + p/n < \gamma < p$ as $1 < p < n$. But in the latter case,

*The work supported in part by Zhongshan University Science Research Fund and by the Foundation of Zhongshan University Advanced Research Centre.

in order to make (0.1) have a meaning it is needed to add restrictions to u. In what follows we require $u \in \overset{\circ}{W}{}^{1,p}(G) \cap L^t(G)$ with t satisfying (1.7) and corresponsively $K(\psi)$ is replaced by $K_t(\psi) = \{u \mid u \in \overset{\circ}{W}{}^{1,p}(G) \cap L^t(G) + u_0 \subset W^{1,p}(G) \cap L^t(G)$ and $u(x) \geq \psi(x)$ for a.e. x in $G\}$ in the case $p - 1 + p/n < \gamma < p$.

1. Boundedness

Let $\underline{A}(x, u, \xi)$ and $B(x, u, \xi)$ be defined on $G \times E^1 \times E^n$, continuous in u and ξ for fixed x, and measurable in x for fixed u and ξ, respectively. In addition $\underline{A}$ and B satisfy the following structural conditions:

$$\xi \cdot \underline{A}(x, u, \xi) \geq k_0|\xi|^p - k|u|^q - f_0(x) \tag{1.1}$$

$$|\underline{A}(x, u, \xi)| \leq k_1|\xi|^{p-1} + k|u|^{q/p'} + f_1(x) \tag{1.2}$$

$$|B(x, u, \xi)| \leq c(x)|\xi|^\gamma + k|u|^{q-1} + f_2(x) \tag{1.3}$$

where the constants $k_1 \geq k_0 > 0$, $p > 1$, $1/p' + 1/p = 1$, $k > 0$, $p - 1 \leq \gamma < p$, $q = p^* = np/(n-p)$ as $1 < p < n$ and $p \leq q < +\infty$ as $p = n$.

$$f_i(x) \in L^{s_i}(G) \ (i = 0, 1, 2), \ s_0, s_2 > n/p \text{ and } s_1 > n/(p-1) \tag{1.4}$$

$$c(x) \in L^{r_1}(G) \tag{1.5}$$

$$\begin{cases} r_1 = n & \text{as } 1 < p < n \text{ and } \gamma = p - 1 \\ 1/r_1 = 1 - 1/p - 1/p^* & \text{as } 1 < p < n \text{ and } p - 1 < \gamma < p - 1 + p/n \\ r_1 = +\infty & \text{as } 1 < p < n \text{ and } \gamma = p - 1 + p/n \\ r_1 > n/(p-\gamma) & \text{as } 1 < p < n \text{ and } p - 1 + p/n < \gamma < p \\ r_1 > p/(p-\gamma) & \text{as } p = n \text{ and } p - 1 \leq \gamma < p \end{cases} \tag{1.6}$$

Let t satisfy

$$(\gamma + 1 - p)/t + (p - \gamma)/q + \gamma/p + 1/r_1 = 1 \text{ as } 1 < p < n \tag{1.7}$$

Theorem 1 *Suppose $1 < p \leq n$ and the conditions (1.1)–(1.7) are fulfilled. Suppose the solution of (0.1), u, satisfies*

$$\begin{array}{l} u \in K(\psi) \text{ as } 1 < p < n \text{ and } p - 1 \leq \gamma \leq p - 1 + p/n \text{ or } p = n \text{ and } p - 1 \leq \gamma < p \\ u \in K_t(\psi) \text{ as } 1 < p < n \text{ and } p - 1 + p/n < \gamma < p \end{array} \tag{1.8}$$

If ψ is bounded locally in G, then, so is u.

Proof Denote $B(x, r) = \{|y - x| < r\}$ and $B(0, r) = B(r)$. Suppose $B(2\rho) \subset\subset G$ and $\rho \leq \rho_1 < \rho_0 \leq 2\rho$. Let $\zeta(x) = \zeta(|x|)$ be a piecewise linear continuous function of $|x|$ satisfying $\zeta(x) = 1$ as $|x| \leq \rho_1$ and $\zeta(x) = 0$ as $|x| \geq \rho_0$. Take $k \geq k^* = \|\psi\|_{L^\infty(B(2\rho))}$ and $\tau = p^2/(p-1)$. Then, $v = u - \zeta^\tau(u - k)^+$ may be taken as a test function. Inserting it into (0.1) we obtain that

$$0 \geq \int_{A(k,\rho_0)} \zeta^\tau |\nabla u|^p dx - \int_{A(k,\rho_0)} \{\zeta^\tau(k|u|^q + f_0(x))$$