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EXISTENCE OF SOLUTIONS FOR QUASILINEAR WEAKLY COERCIVE ELLIPTIC VARIATIONAL INEQUALITIES*

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Abstract In this note we give an existence result to a class of variational inequalities associated with quasilinear elliptic operators of second order with lower order terms. We prove "*a priori*" estimate by an extension of the truncation method to the nonlinear case.

Key Words Weakly coercive; variational inequality; truncation method; existence; closed convex set.

Classification 35A05, 35J85, 35R45.

1. Introduction

In this note, we consider the variational inequality

$$u \in \boldsymbol{K}, \quad \langle Qu, v-u\rangle \geq 0, \quad \forall v \in \boldsymbol{K} \tag{1}$$

where $\boldsymbol{K}$ is a convex, closed, nonempty set in the Banach space $\boldsymbol{C}$, $W_0^{1,p}(\Omega) \subseteqq \boldsymbol{B} \subsetneqq W^{1,p}(\Omega)$, $1 \leq p < \infty$, which is defined on an open bounded domain $\Omega \subset \boldsymbol{R}^N$, $N \ngeq 2$, and

$$\langle Qu, v\rangle = \int_\Omega A_i(x,u,Du)D_i v dx + \int_\Omega A_0(x,u,Du)v dx, \quad u,v \in \boldsymbol{B} \tag{2}$$

where A_i $(i = 1,\cdots,N)$ and A_0 are Carathéodory functions: $\Omega \times \boldsymbol{R} \times \boldsymbol{R}^N \to \boldsymbol{R}$ and $Q : \boldsymbol{B} \to \boldsymbol{B}'$. We denote $Du = (D_1 u, \cdots, D_N u)$ and we use the sum convention on $i = 1,\cdots,N$. We consider the existence of the solution of this problem. The structure of the quasilinear elliptic operator Q is assumed as follows:

(A1) Weakly coercive assumption:

$$A_i(x,u,\xi)\xi_i \geq \alpha|\xi|^p - [f(x) + e(x)|u|^\theta]|\xi|$$

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for a.e. $x \in \Omega$, $u \in \boldsymbol{R}$ and $\xi \in \boldsymbol{R}^N$, where $\alpha > 0$ is a constant, $\theta \in [0, p-1)$, $e \in L^{\delta}(\Omega)$, with $\delta = p'$, when $p > N$; $\delta > p'$, when $p = N$; $\delta = \dfrac{pN}{pN - (N-p)\theta - N}$, when $p < N$, $f \in L^{p'}(\Omega)$. This is weaker than the strong coercive condition, used for instance in [1]–[3] or [4].

(A2) Growth assumption on the high order term:

$$\sum_{i=1}^{N} |A_i(x, u, \xi)| \le C|\xi|^{p-1} + e(x)|u|^{\theta} + f(x)$$

for a.e. $x \in \Omega$, $u \in \boldsymbol{R}$ and $\xi \in \boldsymbol{R}^N$, where C is a positive constant, θ, $e(x)$ and $f(x)$ are as in (A1).

(A3) Growth assumption on the lower order term:

$$|A_0(x, u, \xi)| \le b_0(x)|\xi|^{p-1} + e_0(x)|u|^{\theta_0} + f_0(x)$$

for a.e. $x \in \Omega$, $u \in \boldsymbol{R}$ and $\xi \in \boldsymbol{R}^N$, where $\theta_0 \in \left[0, \dfrac{pN}{N-p}\right]$, $b_0 \in L^{\lambda}(\Omega)$, with $\lambda = N$, when $p < N$; $p < \lambda < \infty$, when $p = N$; $\lambda = p$, when $p > N$, $e_0 \in L^{\sigma}(\Omega)$, with $\sigma = 1$, when $p > N$; $\sigma > 1$, when $p = N$; $\sigma = \dfrac{pN}{pN - (\theta_0 + 1)(N-p)}$, when $p < N$, $f_0 \in L^{p^{*\prime}}(\Omega)$, where $\dfrac{1}{p^{*\prime}} + \dfrac{1}{p^*} = 1$, with $p^* = \infty$, when $p > N$; $p^* = s$ for all $s > 1$, when $p = N$; and $p^* = \dfrac{pN}{N-p}$, when $p < N$.

(A4) Monotonicity assumption on the high order term:

$$[A_i(x, u, \xi) - A_i(x, u, \eta)](\xi_i - \eta_i) > 0$$

for a.e. $x \in \Omega$, $u \in \boldsymbol{R}$ and $\xi \in \boldsymbol{R}^N$.

(A5) Monotonicity assumption on the lower order term:

$$[A_0(x, u, \xi) - A_0(x, v, \xi)](u - v) \ge 0$$

for a.e. $x \in \Omega$, $u \in \boldsymbol{R}$ and $\xi \in \boldsymbol{R}^N$.

Remark 1 If $\|b_0\|_\lambda$ is small enough, the operator satisfies the coercive condition in classical sense, and the result is well known. We are interested here in studying the case when $\|b_0\|_\lambda$ may be large, so that the coercive condition fails in classical sense.

Remark 2 As an example of an operator satisfying these assumptions we may give

$$Qu = -D_i[\alpha_i|Du|^{p-2}D_i u + e_i(x)|u|^{\theta} + f_i(x)] + b_i(x)|Du|^{p-2}D_i u + e_0|u|^{\theta_0 - 1}u + f_0(x)$$

The case of $p = 2$ for the variational inequalities of obstacle type has been discussed in [5] for linear operators and in [6] for quasilinear ones. In the equation case, whe