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THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF A CLASS OF EVOLUTION EQUATIONS

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Abstract In this paper, a class of evolution equations which have special boundary conditions is discussed. The spectrum properties of an operator which is associated with this class of equations are provided, and the existence and uniqueness of solutions of this class of equations is proved by the theory of perturbation of C_0-semigroups. In the last part of this paper, a condition which ensures the solutions to be non-negative is given. Some results of the time-independent population systems in [1–5] are our examples.

Key Words Evolution equation; C_0-semigroup.
Classification 92D25, 35E05.

1. Introduction

Let $M > 0$, N be a natural number, and $H = \underbrace{L^2[0, M] \times \cdots \times L^2[0, M]}_{N}$ with inner product defined by

$$(\underset{\sim}{x}, \underset{\sim}{y}) = \sum_{k=1}^{N} \int_0^M x_k(s)\overline{y_k(s)}ds$$

where $\underset{\sim}{x}(s) = (x_1(s), \cdots, x_N(s))^T$, $\underset{\sim}{y}(s) = (y_1(s), \cdots, y_N(s))^T \in H$. Then H is a Hilbert space. We consider the following first order evolution equation in H:

$$\begin{cases} \dfrac{\partial \underset{\sim}{x}(t,s)}{\partial t} + \dfrac{\partial \underset{\sim}{x}(t,s)}{\partial s} = -\Lambda(s)\underset{\sim}{x}(t,s) \\ \underset{\sim}{x}(t,0) = \displaystyle\int_0^M \phi(s)\underset{\sim}{x}(t,s)ds \\ \underset{\sim}{x}(0,s) = \underset{\sim}{x}^{(0)}(s) \end{cases} \tag{1}$$

where $\underset{\sim}{x}(t,s) = (x_1(t,s), \cdots, x_N(t,s))^T$, $\underset{\sim}{x}^{(0)}(s) = (x_1^{(0)}(s), \cdots, x_N^{(0)}(s))^T$

$$\Lambda(s) = \begin{pmatrix} \lambda_{11}(s) \cdots \lambda_{1N}(s) \\ \cdots\cdots\cdots \\ \lambda_{N1}(s) \cdots \lambda_{NN}(s) \end{pmatrix}, \quad \phi(s) = \begin{pmatrix} \varphi_{11}(s) \cdots \varphi_{1N}(s) \\ \cdots\cdots\cdots \\ \varphi_{N1}(s) \cdots \varphi_{NN}(s) \end{pmatrix}$$

$\lambda_{ij}(s), \varphi_{ij}(s)$ $(i, j = 1, \cdots, N)$ are measurable functions. Defining the operators A_0 and A_1 in the state space H respectively by

$$\begin{cases} D(A_0) = \left\{ \underline{x}(s) \middle| \underline{x}(s), -\dfrac{d\underline{x}(s)}{ds} \in H \quad \text{and } \underline{x}(0) = \displaystyle\int_0^M \phi(s)\underline{x}(s)ds \right\} \\ A_0\underline{x}(s) = -\dfrac{d\underline{x}(s)}{ds} \quad \text{for } \underline{x}(s) \in D(A_0) \end{cases}$$

and

$$A_1\underline{x}(s) = -\Lambda(s)\underline{x}(s) \quad \text{for} \quad \underline{x}(s) \in H$$

we can write Equation (1) as the following abstract evolution equation in H:

$$\begin{cases} \dfrac{d\underline{x}(t)}{dt} = (A_0 + A_1)\underline{x}(t) \\ \underline{x}(0) = \underline{x}^{(0)}(s) \end{cases} \tag{2}$$

As two different main models in population theory, the time-independent population evolution equation ([4]) and the parity progressive population evolution equation ([5]) are both examples of Equation (1). One of the purposes of this paper is to uncover the inherent connection between the two population evolution equations. We also expect the applications of Equation (1) in other fields.

We first give the spectrum properties of operator A_0, then prove that A_0 generates a C_0-semigroup and obtain the existence and uniqueness of solutions of Equation (2) (i.e. Equation (1)) by the theory of perturbation theory of C_0-semigroups. Finally, we prove that for every non-negative $\underline{x}^{(0)}(s) \in D(A_0)$ the solution of Equation (2) (i.e. Equation (1)) is non-negative if $\lambda_{ij}(s) \le 0$ for $i \ne j$ and $\varphi_{ij}(s)$ $(i, j = 1, \cdots, N)$ are non-negative.

2. The Spectrum Properties of Operator A_0

Theorem 1 *Let λ be a complex number, and*

$$\Delta(\lambda) = \begin{pmatrix} 1-\delta_{11}(\lambda) & -\delta_{12}(\lambda) & \cdots & -\delta_{1N}(\lambda) \\ -\delta_{21}(\lambda) & 1-\delta_{22}(\lambda) & \cdots & -\delta_{2N}(\lambda) \\ \cdots & \cdots & \cdots & \cdots \\ -\delta_{N1}(\lambda) & -\delta_{N2}(\lambda) & \cdots & 1-\delta_{NN}(\lambda) \end{pmatrix}$$

where $\delta_{ij}(\lambda) = \displaystyle\int_0^M \varphi_{ij}(s)e^{-\lambda s}ds$ $(i, j = 1, \cdots, N)$. Let $F(\lambda)$ be the determinant of $\Delta(\lambda)$, then the following conclusions hold:

(a) *If $F(\lambda) \ne 0$ then $\lambda \in \rho(A_0)$ (the resolvent set of operator A_0) and the resolvent operator $R(\lambda, A_0)$ is compact;*

(b) *If $F(\lambda) = 0$, then $\lambda \in \sigma_p(A_0)$ (the eigenvalues set of operator A_0) and the geometric multiplicity of λ is N-rank $(\Delta(\lambda))$;*