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LOCAL BOUNDEDNESS OF MINIMIZERS OF ANISOTROPIC FUNCTIONALS

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Abstract Using the theory of anisotropic Sobolev spaces, we discuss in this paper the relation between the growth conditions and the local boundedness of minimizers of an anisotropic variational problem. This thoroughly explains the counterexample due to Giaquinta (1987). In the sense of local boundedness, we point out a critical index.

Key Words Variational problem; anisotropic Sobolev spaces; local boundedness; minimizers.

Classification 35B25, 35J55.

0. Introduction

For the relation between the growth conditions and the local boundedness of minimizers of a variational problem, Giaquinta M. [1] proposed an interesting counterexample on finding minimizer for

$$\mathcal{F}[u]=\int_B\left[\sum_{i=1}^{n-1}\left(\frac{\partial u}{\partial x_i}\right)^2+\frac{1}{2}\left(\frac{\partial u}{\partial x_n}\right)^4\right]dx \tag{0.1}$$

where B is a unit sphere in R^n with center at 0. Obviously, the integrand in (0.1) satisfies

$$c_0(|p|^m-1)\le F(x,u,p)\le c_1(|p|^m+1),\ c_0,c_1>0,\ m>1 \tag{0.2}$$

But it can be verified that function

$$u(x)=\sqrt{\frac{n-4}{24}}\cdot\frac{x_n^2}{r},\quad r=\left(\sum_{i=1}^{n-1}x_i^2\right)^{1/2} \tag{0.3}$$

is a minimizer for $n\ge 6$ which is not locally bounded in B.

In fact, this is a special anisotropic variational problem. Hong Minchun [2] explained this phenomenon to certain extent; he proved that if $F(x,u,p)$ satisfies

$$c_0(|\widetilde{p}|^{m_1}+|\widetilde{\widetilde{p}}|^{m_2}-1)\le F(x,u,p)\le c_1(|\widetilde{p}|^{m_1}+|\widetilde{\widetilde{p}}|^{m_2}+1) \tag{0.4}$$

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where $\tilde{p}$ and $\tilde{\tilde{p}}$ denote, respectively, $(p_1,\cdots,p_{n_1})$ and $(p_{n_1+1},\cdots,p_n)$; $n_1<n$, $1\le m_1\le m_2<\overline{m}_1=\dfrac{nm_1}{n-m_1}$, $m_1<n$, then for the local minimizer $u\in W^{1,m_2}_{\rm loc}(\Omega)$ of $\mathcal{F}[u]$, u must be locally bounded in Ω.

In Giaquinta's counterexample, $\Omega=B$ and $m_1=2$, $m_2=4$, $n_1=n-1$. Therefore, $m_2<\overline{m}_1$ is not satisfied for $n\ge 6$.

It is then natural to ask that for the same question as in [2], whether we can improve the condition $m_2<\overline{m}_1$ or not? Further, we want to know if the structure condition is changed into

$$\lambda\sum_{i=1}^{n}|\eta_i|^{p_i}-C\le F(x,u,\eta)\le\mu\sum_{i=1}^{n}|\eta_i|^{p_i}+C \tag{0.5}$$

where $0<\lambda\le\mu$, $p_i\ge 1$ $(i=1,2,\cdots,n)$, then in order to assure the local boundedness of the minimizer, what conditions should we impose on $\{p_i\}_{i=1}^n$ so that once this condition is violated then there is a counterexample? On the other hand, to what space should the minimizer belong? This paper is to answer these questions.

1. Main Result

The main result is the following theorem

Theorem *Let $\mathcal{F}[u,\Omega]=\int_\Omega F(x,u,Du)dx$ satisfy* (0.5) *and*

$$\sum_{i=1}^{n}\frac{1}{p_i}>1,\quad \max(p_1,\cdots,p_n)<q=n\Big(\sum_{i=1}^{n}\frac{1}{p_i}-1\Big)^{-1} \tag{1.1}$$

Then the minimizer $u(x)$ of $\mathcal{F}[u,\Omega]$ in $W^{1,(p_i)}_{\rm loc}(\Omega)$ is locally bounded in Ω.

Remarks

(1) See [3] for the detail discussions on anisotropic Sobolev spaces $W^{1,(p_i)}(\Omega)$.

(2) By a local minimizer u of $\mathcal{F}[u,\Omega]$ we mean that $u\in W^{1,(p_i)}_{\rm loc}(\Omega)$ and for any $\varphi\in \overset{\circ}{W}{}^{1,(p_i)}(\Omega)$

$$\mathcal{F}[u,\mathrm{supp}\varphi]\le\mathcal{F}[u+\varphi,\mathrm{supp}\varphi] \tag{1.2}$$

under condition (0.5).

When $p_1=\cdots=p_{n_1}=m_1$, $p_{n_1+1}=\cdots=p_n=m_2$ it goes back to the problem of [2]. We should point out that it is inappropriate to find minimizer in $W^{1,m_2}_{\rm loc}(\Omega)$ as in [2]. In Giaquinta's counterexample, $p_1=\cdots=p_{n-1}=2$, $p_n=4$, then it verifies that (1.1) is satisfied if and only if $n\le 5$. Moreover, the improvement of [2] is $m_2<\dfrac{n_1m_1}{n_1-m_1}$. What we should clarify here for $\sum_{i=1}^{n}\dfrac{1}{p_i}<1$ is that $W^{1,(p_i)}(\Omega)$ is embedded into anisotropic Hölder space (see [3]) so that local boundedness automatically holds. Furthermore, (1.1) is satisfied under $m_1\ge n_1$, so we improve the condition of [2] to $m_2<\dfrac{n_1m_1}{n_1-m_1}$.