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THRESHOLD RESULTS FOR SEMILINEAR PARABOLIC EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS*

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Abstract This paper deals with the following semilinear parabolic equations with nonlinear boundary conditions

$$u_t - \Delta u = f(u) - \lambda u, x \in \Omega,\ t > 0$$
$$\frac{\partial u}{\partial n} = g(u), \qquad x \in \partial\Omega,\ t > 0$$

It is proved that every positive equilibrium solution is a threshold.
Key Words Nonlinear boundary conditions; threshold results; upper and lower solutions.
Classification 35K60, 35B40, 35K20.

1. Introduction and Main Results

This paper discusses the initial boundary value problem

$$\begin{cases} u_t - \Delta u = f(u) - \lambda u, & x \in \Omega,\ t > 0 \\ \dfrac{\partial u}{\partial n} = g(u), & x \in \partial\Omega,\ t > 0 \\ u(x,0) = u_0(x) \geq 0, & x \in \overline{\Omega} \end{cases} \tag{1.1}$$

where $\Omega \subset \boldsymbol{R}^N$ is a bounded domain with C^2 boundary $\partial\Omega$, n is the outward normal on $\partial\Omega$, $\lambda > 0$, $u_0(x) \in L^\infty(\Omega)$ is a nonnegative function, $f(y), g(y) \in C^1([0,+\infty))$ and satisfy

(H) $\lim\limits_{y\to+\infty} f(y)/y > \lambda$, $\int^{+\infty} \dfrac{dy}{g(y)} < +\infty$ and $f(ly) > lf(y)$, $g(ly) \geq lg(y)$ for any $l > 1$ and $y > 0$. Moreover there exists $M_0 > 0$ such that $g(y) > 0$, $g'(y) \geq 0$ for $y \geq M_0$.

The stationary problem of (1.1) is

$$\begin{cases} -\Delta u = f(u) - \lambda u, & x \in \Omega \\ \dfrac{\partial u}{\partial n} = g(u), & x \in \partial\Omega \end{cases} \tag{1.2}$$

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For some special functions $f(u)$ and $g(u)$, (1.2) has a positive classical solution. For example, choose $f(u) = u^p$, $g(u) = u^q$ with $1 < p \leq (N+2)/(N-2)$, $1 < q \leq N/(N-2)$ and $N \geq 3$, the assumption (H) holds. Using the results and methods of [1] we have that (1.2) has a positive solution $\overline{u}(x) \in H^1(\Omega)$. By the methods of [2, 3] we can prove that $\overline{u}(x)$ is a classical solution of (1.2), i.e. $\overline{u}(x) \in C^1(\overline{\Omega}) \cap C^2(\Omega) \cap W^2_m(\Omega)$ for some $m > N$.

Our main aim is to prove that every positive classical solution of (1.2) is a threshold. That is the following theorem.

Theorem 1.1 *Suppose that* (H) *holds and* $\overline{u}(x)$ *is a positive classical solution of* (1.2). *We have*

(i) *If* $u_0(x) \geq \overline{u}(x)$ *and* $u_0(x) \not\equiv \overline{u}(x)$, *then the solution of* (1.1) *blows up in finite time.*

(ii) *If* $0 \leq u_0(x) \leq \overline{u}(x)$, *then the solution of* (1.1) *exists globally. Moreover, if in addition* $u_0(x) \not\equiv \overline{u}(x)$, *then the global solution* $u(x,t)$ *of* (1.1) *satisfies* $\lim\limits_{t\to+\infty} u(x,t) = 0$ *pointwise on* $\overline{\Omega}$.

These results not only show that the positive classical solution of (1.2) is unstable but also show that the ordered positive classical solution of (1.2) is unique. To our knowledge these results are all new.

Define

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\lambda}{2}\int_\Omega u^2 dx - \int_\Omega F(u)dx - \int_{\partial\Omega} G(u)dS_x$$

where $F(u) = \int_0^u f(y)dy$, $G(u) = \int_0^u g(y)dy$.

It was proved in [4, 5] that if $E(u_0) \leq 0$, then the solution of (1.1) blows up in finite time. If we choose $f(u) = u^p$, $g(u) = u^q$ and $u_0(x) = l\overline{u}(x)$ with $p, q > 1$ and $l > 0$, where $\overline{u}(x)$ is a positive classical solution of (1.2), then $F(u) = u^{p+1}/(p+1)$, $G(u) = u^{q+1}/(q+1)$ and

$$\frac{1}{2}\int_\Omega |\nabla \overline{u}|^2 dx - \frac{1}{2}\int_\Omega \overline{u}^{p+1} dx - \frac{1}{2}\int_{\partial\Omega} \overline{u}^{q+1} dS_x + \frac{\lambda}{2}\int_\Omega \overline{u}^2 dx = 0$$

So that

$$\begin{aligned} E(u_0) &= \frac{l^2}{2}\int_\Omega |\nabla \overline{u}|^2 dx - \frac{l^{p+1}}{p+1}\int_\Omega \overline{u}^{p+1} dx - \frac{l^{q+1}}{q+1}\int_{\partial\Omega} \overline{u}^{q+1} dS_x + \frac{l^2\lambda}{2}\int_\Omega \overline{u}^2 dx \\ &= l^2\Big[\Big(\frac{1}{2} - \frac{l^{p-1}}{p+1}\Big)\int_\Omega \overline{u}^{p+1} dx + \Big(\frac{1}{2} - \frac{l^{q-1}}{q+1}\Big)\int_{\partial\Omega} \overline{u}^{q+1} dS_x\Big] > 0 \end{aligned}$$

if l is close to 1. For this case, the results of [4, 5] give nothing about the large time behaviors of the solution of (1.1). But our theorem asserts that the solution of (1.1) blows up in finite time when $l > 1$, exists globally and tends to zero as $t \to +\infty$ when $l < 1$.