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EXISTENCE AND NONUNIQUENESS OF SOLUTIONS TO A ROBIN BOUNDARY PROBLEM FOR SEMILINEAR ELLIPTIC EQUATIONS *

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Abstract Sufficient conditions for existence and nonuniqueness of radially symmetric solutions to the Robin boundary problem of the form

$$\Delta u + a(\|x\|)|u|^{-p} = 0 \qquad \text{in } B \subset R^N$$
$$\frac{\partial u}{\partial \nu} + \lambda u = -\alpha \qquad \text{on } \partial B$$

are obtained.

Key Words Robin boundary problem; positive solution; negative solution; nonuniqueness.

Classification 35J25, 34B15.

1. Introduction

In this paper we consider the existence and nonuniqueness of radially symmetric solutions of the Robin boundary problem

$$\Delta u + a(\|x\|)|u|^{-p} = 0 \qquad \text{in } B \subset R^N \tag{1.1}$$
$$\frac{\partial u}{\partial \nu} + \lambda u = -\alpha \qquad \text{on } \partial B \tag{1.2}$$

where B is an open unit ball centered at the origin and ∂B is its boundary, $\|x\|$ is the Euclidean norm of x, p, λ and α are all positive constants, $N \geq 3$, ν is the outward unit normal on ∂B, and $a(t)$ is a continuous function defined on $[0,1]$ such that $a(t) > 0$ a.e. in $[0,1]$.

Equation (1.1) arises in mathematical physics, and the existence and uniqueness of positive solutions to the Dirichlet problem for (1.1) have been investigated by several authors [1–5]. Moreover, most of the available literature on determining the existence of positive radially symmetric solutions to Equation (1.1) (or more general form), for examples [6–9] and their references, studied the case $p < 0$.

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The search for radially symmetric solutions of the problem (1.1), (1.2) leads to the following problem in ordinary differential equations:

$$y''(t) + \frac{N-1}{t}y'(t) + a(t)|y|^{-p} = 0, \quad 0 < t < 1 \tag{1.3}$$

$$y'(0) = 0,\ y'(1) + \lambda y(1) = -\alpha \tag{1.4}$$

which is the problem to be studied in this paper. Actually, we will prove that the problem (1.3), (1.4) has a positive solution and a negative solution under certain conditions.

2. Positive Solution

In this section we consider the existence of positive solutions to the problem (1.3), (1.4), namely

$$y''(t) + \frac{N-1}{t}y'(t) + a(t)y^{-p} = 0, \quad 0 < t < 1 \tag{2.1}$$

$$y'(0) = 0,\ y'(1) + \lambda y(1) = -\alpha \tag{2.2}$$

Instead of solving (2.1), (2.2) directly, we consider the following boundary value problem

$$y''(t) + \frac{N-1}{t}y'(t) + a(t)y^{-p} = 0, \quad 0 < t < 1 \tag{2.1}$$

$$y'(0) = 0,\ y(1) = \beta \geq 0 \tag{2.3}$$

By a positive solution of (2.1), (2.3) we mean a function $y(t)$ in $C[0,1] \cap C^1[0,1) \cap C^2(0,1)$ which satisfies (2.1) and (2.3), and is positive in the interval $[0,1)$.

It is easy to see that the problem (2.1), (2.3) is equivalent to the following integral equation

$$y(t) = \beta + \int_0^1 G(t,s)a(s)[y(s)]^{-p}\mathrm{d}s \text{ on } [0,1] \tag{2.4}$$

where

$$G(t,s) = \begin{cases} \frac{1}{N-2}(t^{2-N}-1)s^{N-1} & \text{for } 0 \leq s \leq t \\ \frac{1}{N-2}(s^{2-N}-1)s^{N-1} & \text{for } t \leq s \leq 1 \end{cases}$$

is positive in $(0,1) \times (0,1)$.

It follows from (2.4) that

$$y'(t) = -t^{1-N}\int_0^t s^{N-1}a(s)[y(s)]^{-p}\mathrm{d}s \quad \text{on } (0,1] \tag{2.5}$$

Moreover, for any subinterval $[a,b] \subset [0,1]$, $y(t)$ can be written as

$$y(t) = \int_a^b G^{ab}(t,s)a(s)[y(s)]^{-p}\mathrm{d}s + E^{ab}(t) \quad \text{on } [a,b] \tag{2.6}$$