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THE EXISTENCE AND UNIQUENESS OF THE WEAK SOLUTION FOR THE EVOLUTIONARY ELECTROCHEMICAL MACHINING PROBLEM *

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Abstract A time dependent electrochemical machining problem, in which the cathode is fed towards the anode with a constant velocity, is studied. We prove the existence and uniqueness of the weak solution for the problem under the assumption that the cathode is $C^{1+\beta}$ for some $\beta \in (0,1)$.
Key Words Evolutionary elliptic free boundary problem; electrochemical machining problem; existence and uniqueness.
Classification 35J.

1. Introduction

Electrochemical machining (ECM) is a process used to machine a metal workpiece (the anode) by a metal tool piece (the cathode). The two electrodes are separated by a small gap through which the flow of electrolyte solutions is pumped. As applying potential difference, an electric current flows across the gap and the metal is removed electrochemically on the anode surface, while the cathode is unaffected by the process. As electrolyte dissolution proceeds, the cathode-tool can be fed mechanically towards the anode-workpiece in order to maintain the machining action. Here we shall assume that it is moved with constant speed in the positive y axis direction. Denote by $\Phi = \Phi(x, y, t)$ the potential at time $t > 0$. Then, in a coordinate system for which the cathode is stationary, ECM model under consideration is as follows:

$$-\Delta\Phi = 0 \quad \text{in electrolyte} \tag{1.1}$$

$$\Phi = 0 \quad \text{at cathode} \tag{1.2}$$

$$\Phi = 1 \quad \text{at anode} \tag{1.3}$$

$$V_n = \frac{\partial \Phi}{\partial n} - \cos(n, y) \quad \text{on anode surface} \tag{1.4}$$

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where the anode surface is a free boundary, n is the outward normal to the anode surface, and V_n is the velocity of the free boundary. About the (ECM) mathematical model we refer to [1] [2] for more details. The problem (1.1)-(1.4) is a kind of evolutionary elliptic free boundary problem, in which the free boundary changes with time and at each fixed time the unknown function satisfies a boundary value problem for an elliptic equation. Clearly it is related to the one phase Stefan problem with zero specific heat. In [3] a numerical procedure for calculating the solution of (1.1)-(1.4) was derived. Note that (1.1) and (1.4) with certain boundary conditions describes the time dependent and general form dam problem with the incompressible fluid. In [4] the existence of the weak solution for the time dependent dam problem with general form was proved and the uniqueness of the weak solution in the case of the incompressible fluid was proved only for the rectangular dam. The domain of finding the solution for the dam problem is naturally the dam itself, while we need to determine the domain of finding solution such that the free boundary doesn't intersect the boundary of the domain for the problem (1.1)-(1.4). Moreover we shall prove the uniqueness of the weak solution of (1.1)-(1.4), though it corresponds to the case of incompressible fluid in the dam problem.

In Section 2 the weak formulation of the problem and our main result are stated. By using parabolic regularization the existence of the weak solution is proved in Section 3. And we shall show, in Section 4, the uniqueness of the weak solution by modifying the Kamin's classical method [5], which is used to prove the uniqueness of the Stefan problem with positive specific heat, and by using the uniform boundary estimate of gradient of the solution for the linear parabolic equations with the boundary being in $C^{1+\beta}$ $(0<\beta<1)$.

2. The Weak Formulation and the Main Result

For simplicity we consider two dimension problem in a domain, which is periodic along x axis direction. Let L be a given positive constant. The cathode τ is given by $y=c(x)$ satisfying

$$\begin{aligned} &c(x)\in C^{1+\beta}(\mathcal{R}^1) \quad \text{for some } \beta\in(0,1)\\ &c(x+L)=c(x)\geq 0 \quad \text{for all } x\in\mathcal{R}^1 \end{aligned} \tag{2.1}$$

The initial anode Γ_0 is $y=a_0(x)$ satisfying

$$a_0(x)\in C^1(\mathcal{R}^1),\ a_0(x+L)=a_0(x)>c(x) \quad \text{for all } x\in\mathcal{R}^1 \tag{2.2}$$

At each time t, denote the anode Γ_t by $y=a(x,t)$, and the region occupied by the electrolyte by

$$\Omega_t\equiv\{(x,y); x\in\mathcal{R}^1, c(x)<y<a(x,t)\} \tag{2.3}$$