J.Partial Diff. Eqs. 8(1995), 341–350 Vol.8, No.4

L^p-L^q ESTIMATES FOR A LINEAR PERTURBED KLEIN-GORDON EQUATION

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(Received Nov. 4, 1993; revised Oct. 5, 1994)

Abstract We consider L^p-L^q estimates for the solution $u(t,x)$ to the following perturbed Klein-Gordon equation

$$\begin{aligned} &\partial_{tt}u - \Delta u + u + V(x)u = 0 \qquad x \in R^n, n \geq 3 \\ &u(x,0) = 0, \quad \partial_t u(x,0) = f(x) \end{aligned}$$

We assume that the potential $V(x)$ and the initial data $f(x)$ are compact, and $V(x)$ is sufficiently small, then the solution $u(t,x)$ of the above problem satisfies

$$\|u(t)\|_q \leq Ct^{-a}\|f\|_p \quad \text{for } t > 1$$

where a is the piecewise-linear function of $1/p$ and $1/q$.
Key Words L^p-L^q estimates; Klein-Gordon equation; perturbed potential.
Classification 35L05, 35B20, 35B45.

1. Introduction

We consider L^p-L^q estimates for the solution $u(t,x)$ to the following perturbed Klein-Gordon equation

$$\begin{aligned} &\partial_{tt}u - \Delta u + u + V(x)u = 0 \qquad x \in R^n, n \geq 3 \\ &u(x,0) = 0, \quad \partial_t u(x,0) = f(x) \end{aligned} \tag{1.1}$$

In the unperturbed case ($V = 0$), the mapping $f \to T_t f = u(t)$ is given by the operator corresponding to the Fourier multiplier $\sin t\langle\xi\rangle / \langle\xi\rangle$, i.e.

$$T_t f = \int e^{ix\cdot\xi}\frac{\sin t\langle\xi\rangle}{\langle\xi\rangle}\hat{f}(\xi)d\xi, \quad \langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}} \tag{1.2}$$

Concerning the problem (1.1) Marshall-Strauss-Wainger [1] obtain the following results

Theorem 1.1 (see [1]) *For $n \geq 3$, $V(x) = 0$, let $\left(\frac{1}{p}, \frac{1}{q}\right) \in \Upsilon$, Υ is the closed triangle $P_1P_2P_3$, $f(x) \in L^p(R^n)$. Let $u(t,x)$ be the solution of* (1.1), *then*

$$\|u(t)\|_q \leq Ct^{1-n(1/p-1/q)}\|f\|_p \quad \textit{for } 0 < t < 1 \tag{1.3}$$

$$\|u(t)\|_q \leq Ct^{-a}\|f\|_p \qquad for\ t \geq 1 \tag{1.4}$$

where a is the following piecewise-linear function of $1/p$ and $1/q$,

$$\begin{aligned} a &= n(1-1/p)-(n-2)/q && in\ triangle\ P_1P_5P_6 \\ a &= n/q-(n-1)(1-1/p) && in\ triangle\ P_1P_4P_6 \\ a &= n/2-n/q && in\ quadrilateral\ P_0P_3P_5P_6 \\ a &= -n/2+n/p && in\ quadrilateral\ P_0P_2P_4P_6 \end{aligned} \tag{1.5}$$

Here we define the points

$$P_0 = \Big(\frac{1}{2}, \frac{1}{2}\Big)$$

$$P_1 = \Big(\frac{1}{2}+\frac{1}{n+1}, \frac{1}{2}-\frac{1}{n+1}\Big)$$

$$P_2 = \Big(\frac{1}{2}-\frac{1}{n+1}, \frac{1}{2}-\frac{1}{n-1}\Big)$$

$$P_3 = \Big(\frac{1}{2}+\frac{1}{n-1}, \frac{1}{2}+\frac{1}{n-1}\Big)$$

$$P_4 = \Big(\frac{1}{2}, \frac{1}{2}-\frac{1}{n}\Big)$$

$$P_5 = \Big(\frac{1}{2}+\frac{1}{n}, \frac{1}{2}\Big)$$

$$P_6 = \Big(\frac{1}{2}+\frac{1}{n+2}, \frac{1}{2}-\frac{1}{n+2}\Big)$$

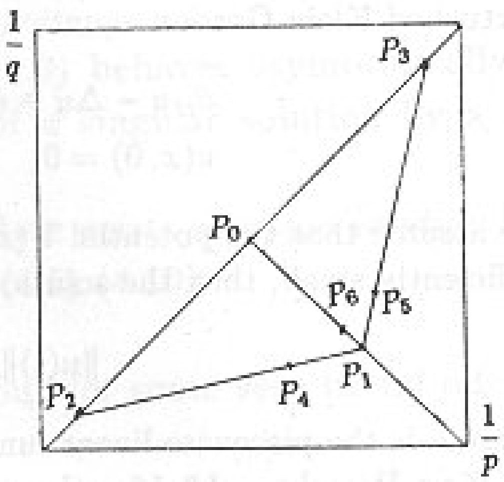

Figure 1

See Figure 1.

Remark 1 On the line of duality P_0P_1, by (1.5) we know that $0 < n(1/p-1/p')-1 \leq a$, therefore for $0 < t < 1$ the estimate (1.3) may be denoted as (1.4). At the point P_5, $a = n(1/p-1/q)-1 = 0$, (1.3) (1.4) are the same. At the points P_3 and P_2, $a = -n/(n-1) < -1$, for $0 < t < 1$ the estimate (1.3) can not be denoted as (1.4).

In perturbed case ($V \neq 0$), Beals and Strauss [2] consider L^p-$L^{p'}$ estimate for the solution to (1.1) at the point P_6 under appropriate conditions on the potential $V(x)$ and the initial data $f(x)$, but they do not give out the detial proof. In this article, for $(1/p, 1/q) \in \Upsilon$, we prove that the estimate (1.4) also holds under some assumptions on $V(x)$ and $f(x)$. Our main result is

Theorem 1.2 *For $n \geq 3$, any $(1/p, 1/q) \in \Upsilon$. Let $V(x) \in C_0^\infty(R^n)$, $f(x) \in L^p(R^n)$ with compact support, without loss of generality we assume* $\text{supp}V(x)$, $\text{supp}f(x) \subset \{x \,|\, |x| \leq 1\}$. *Let $u(t,x)$ be the solution to (1.1), if $\|V\|_\infty$ is sufficiently small, then*

$$\|u(t)\|_q \leq Ct^{-a}\|f\|_p \quad for\ t \geq 1 \tag{1.6}$$

where a as defined by (1.5), C depends on $\|V(x)\|_\infty$.

Remark 2 In Theorem 1.2 if we assume that $\text{supp}V(x) \subset K_1$, $\text{supp}f(x) \subset K_2$, and K_1, K_2 are any two different bounded open sets in R^n, thus in the following proof