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GLOBAL SOLUTIONS AND THEIR LARGE-TIME BEHAVIOR OF CAUCHY PROBLEM FOR EQUATIONS OF DEEP WATER TYPE

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Abstract We consider the large time behavior of the global solution of the Cauchy problem for the equation of Benjamin-Ono type. A series of large-time global estimates for Cauchy problem are constructed. By means of the obtained global estimates uniformly for $0 \leq t < \infty$, the attractors of the Cauchy problem for the mentioned nonlinear equations are considered. And also the dimensions of the global attractor are estimated.
Key Words Large time behavior; Benjamin-Ono type equations; global attractor.

Classification 35Q20.

1. Introduction

The equation, which describes the propagation of the internal waves in the stratified fluid with great depth, can be expressed in the form [1–7]

$$u_t + 2uu_x + Hu_{xx} = 0 \tag{1}$$

where H is the Hilbert transform

$$Hu(x,t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(y,t)}{y-x} dy \tag{2}$$

and P denotes the principle value of the integral. The equation (1) of deep water is also called Benjamin-Ono equation. If the effect of the amplitude of the internal waves is taken into account in the deep fluid, the equation (1) has an additional linear term as follows

$$u_t + c_0 u_x + 2uu_x + Hu_{xx} = 0 \tag{3}$$

There are many investigations of the physical purpose for the nonlinear partial differential equation (1) with singular integral term of deep water. The Backlund transformations, the conservation laws, various soliton solutions and their interactions for the Benjamin-Ono equation (1) are studied in [8–12].

In [13] the initial value problem for the nonlinear singular integral-differential equation

$$u_t + 2uu_x + Hu_{xx} + b(x,t)u_x + c(x,t)u = f(x,t) \tag{4}$$

is studied by the method of fixed point and integral *a priori* estimates. The generalized and classical global solutions are obtained. The classical solution of Cauchy problem for the simple Benjamin-Ono equation (1) is also derived in [14] by the method of semigroups. In [15] the global generalized and classical solution are considered again for the original Benjamin-Ono equation (1) in the Hilbert spaces with half order derivatives.

The purpose of this work is to establish the solutions for the Cauchy problem of the general equation

$$u_t + 2uu_x + \alpha Hu_{xx} - \beta Hu_x + \gamma(x,t)Hu + b(x,t)u_x + c(x,t)u = f(x,t) \tag{5}$$

of Benjamin-Ono type, where $\alpha > 0$ and $\beta \geq 0$ are constants. The term $-\beta Hu_x$ of the equation (5) has special character. The change of the coefficient β shows the interesting behavior of solution of the equation (5). This is a nonlinear partial differential equation with singular integral operators. The solutions of the problem for the above equation are approximated by the solutions of the Cauchy problem for the nonlinear parabolic equation

$$u_t - \epsilon u_{xx} + 2uu_x + \alpha Hu_{xx} - \beta Hu_x + \gamma(x,t)Hu + b(x,t)u_x + c(x,t)u = f(x,t) \tag{6}$$

with Hilbert transforms terms, which is obtained by the addition of a diffusion term ϵu_{xx} with small coefficient $\epsilon > 0$ in the equation (5). The solutions of the Cauchy problem (7) for the nonlinear equation (5) are established by the limiting process of the vanishing of diffusion coefficient $\epsilon \to 0$. The convergence speed is estimated in order of $\epsilon > 0$. And then in later part of this work, we are going to consider the large time behavior of the global solutions of the Cauchy problem for the equation of Benjamin-Ono type. A series of large-time global estimates for the solutions of the problems for the nonlinear parabolic equations with Hilbert operators and the corresponding nonlinear equations of Benjamin-Ono type are constructed. By means of these obtained global estimates, the attractors of the Cauchy problems for the mentioned nonlinear equations are considered. And also the dimensions of the global attractor are estimated.

Let us state some fundamental properties of Hilbert transform [8–12], which are used repeatedly in the further investigation as follows.

Lemma 1 *For any $f(x)$ and $g(x) \in L_2(R)$, there are*

⟨1⟩ $H^2 f = -f$,

⟨2⟩ $H(fg) = H(HfHg) + fHg + gHf$,

⟨3⟩ $\int_{-\infty}^{\infty} f(x)Hg(x)dx = -\int_{-\infty}^{\infty} g(x)Hf(x)dx$,

hence

$$\int_{-\infty}^{\infty} g(x)Hg(x)dx = 0$$

Lemma 2 *For any differentiable function $f(x) \in H^s(R)$ $(s = 1, 2)$, there are*