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LOCAL CLASSICAL SOLUTION OF MUSKAT FREE BOUNDARY PROBLEM *

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Abstract In this paper we consider the two-dimensional Muskat free boundary problem: $\Delta u_i(x,t) = 0$ in space-time domain Q_i $(i = 1, 2)$, here t is a parameter. The unknown surface $\Gamma_{\rho T}$ (free boundary) is the common part of the boundaries of Q_1 and Q_2. The free boundary conditions are $u_1(x,t) = u_2(x,t)$ and $-k_1\dfrac{\partial u_1}{\partial n} = -k_2\dfrac{\partial u_2}{\partial n} = V_n$.
If the initial normal velocity of the free boundary is positive, we shall prove the existence of classical solution locally in time and uniqueness by making use of Newton's iteration method.
Key Words Classical solution; Muskat problem; Newton's iteration method
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1. Introduction and Main Result

Muskat problem is a very old open problem, it was proposed by Muskat in 1934 (see [1]). This problem describes the flows of two fluids in porous media, for example, oil and water. In 1987 Jiang had got a weak formulation for this problem ([2]). In 1989 Liang and Jiang researched an approximating Muskat problem (see [3]). But up to now there is not any mathematical result in the existence of weak or classical solutions.

In this paper, we shall prove the existence of classical solution locally in time and uniqueness by use of Newton's iteration method (see [4], Theorem 15.6). The solution is sought as the limit of the sequence

$$x_{n+1} = x_n - [D\mathcal{F}(x_n)]^{-1}\mathcal{F}(x_n)$$

The difficulty is to prove the invertibility of Frechet derivative operator. In order to state and prove our result, we introduce the following function spaces:

Let G be an open set in $\mathbb{R}^n$, $n = 1, 2$, define

$$C_T^{k+\alpha}(\overline{G}) = C^0([0,T]; C^{k+\alpha}(\overline{G})), \quad 0 < \alpha < 1,\ k = 1, 2, \cdots$$
$$\widehat{C}^{k+\alpha}(\overline{G}) = \{v \in C_T^{k+\alpha}(\overline{G}); \partial_t v \in C_T^{k-1+\alpha}(\overline{G})\} \text{ and}$$

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$$|v|_{\widehat{C}^{k+\alpha}(\overline{G})} = |v|_{C_T^{k+\alpha}(\overline{G})} + |\partial_t v|_{C_T^{k-1+\alpha}(\overline{G})}$$

Denote

$$\underset{\circ}{C}{}_T^{k+\alpha}(\overline{G}) = \{v \in C_T^{k+\alpha}(\overline{G}); v(\cdot,0)=0\}$$

$$\underset{\circ}{\widehat{C}}{}_T^{2+\alpha}(\overline{G}) = \{v \in \widehat{C}_T^{2+\alpha}(\overline{G}); v(\cdot,0)=\partial_t v(\cdot,0)=0\}$$

Let Ω be a bounded annular domain in $\mathbb{R}^2$ with $C^{4+\alpha}$ boundary $\partial\Omega = \Gamma^+ \cup \Gamma^-$, Γ^+ is the inside boundary and Γ^- is the outside boundary. $\Gamma_0 \in \Omega$ is the initial position of free boundary such that $\Gamma_0 \cap \Gamma^\pm = \emptyset$ and Γ^+ is inside of Γ_0 and Γ^- is outside of Γ_0. Define that $\Omega^\pm$ is the part between $\Gamma^\pm$ and Γ_0. For the points of the surface Γ_0 we introduce the coordinate ω; we also denote by $x(\omega) \in \Gamma_0$ and $\vec{n}(\omega)$ the unit normal to Γ_0 directed into Ω^+.

Let ν_0 be a given positive number such that the surface $\{x = x(\omega) \pm 2\vec{n}(\omega)\nu,\ 0 < \nu < \nu_0\}$ has no selfintersection and doesn't intersect $\Gamma^\pm$; let $\rho(\omega,t)$ be a function of class $\widehat{C}_T^{2+\alpha}(\Gamma_0)$ such that $\rho(\omega,0)=0$ and $\max|\rho(\omega,t)| \le \nu_0/4$. We denote by $\Omega_{\rho T}^\pm$ the region bounded by the planes $t=0$, $t=T$, surface $\Gamma_T^\pm = \Gamma^\pm \times [0,T]$ and $\Gamma_{\rho T} = \{(x,t); x = x(\omega) + \vec{n}\rho(\omega,t),\ t\in[0,T]\}$.

The Muskat free boundary problem consists in finding the pressure $u^\pm(x,t)$ (of oil and water) and function $\rho(\omega,t)$ defining an *a priori* unknown surface $\Gamma_{\rho T}$ on the basis of the conditions

$$\Delta u^\pm(x,t) = 0 \quad \text{in } \Omega_{\rho T}^\pm \tag{1.1}$$

$$\partial_n u^+(x,t) = g^+(x,t) \quad \text{on } \Gamma_T^+ \tag{1.2}$$

$$u^-(x,t) = g^-(x,t) \quad \text{on } \Gamma_T^- \tag{1.3}$$

$$u^+(x,t) = u^-(x,t) \quad \text{on } \Gamma_{\rho T} \tag{1.4}$$

$$k^+\partial_n u^+ = k^-\partial_n u^- \quad \text{on } \Gamma_{\rho T} \tag{1.5}$$

$$V_n = -k^+\partial_n u^+ \quad \text{on } \Gamma_{\rho T} \tag{1.6}$$

Equation (1.1) is from Darcy's law neglecting gravity, (1.2) and (1.3) are boundary conditions on fixed boundaries in which n is the exterior unit normal to Γ^+, (1.3) represents supply of water. (1.4)–(1.6) are free boundary conditions, in which $k^\pm = \overline{k}^\pm/\mu^\pm$, $\overline{k}^\pm$ are permeabilities and $\mu^\pm$ are viscosity coefficients. (1.5) and (1.6) have the meaning of the law of energy conservation on the unknown boundary $\Gamma_{\rho T}$ and V_n is the velocity of the free boundary in the direction of $\vec{n}$.

We shall assume

$$\Gamma^\pm, \Gamma_0 \in C^{4+\alpha} \text{ with } 0<\alpha<1 \tag{1.7}$$

$$k^\pm \text{ are constants with } k^+ > k^- > 0 \tag{1.8}$$

$$g^+(x,t) \in C_T^{3+\alpha}(\Gamma^+) \text{ and } \partial_t g^+(x,t) \in C_T^{1+\alpha}(\Gamma^+) \tag{1.9}$$

$$g^-(x,t) \in C_T^{4+\alpha}(\Gamma^-) \text{ and } \partial_t g^-(x,t) \in C_T^{2+\alpha}(\Gamma^-) \tag{1.10}$$