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SOLUTION TO NONLINEAR WAVE EQUATION WITH DELTA INITIAL DATA AND ITS SINGULARITY STRUCTURE *

Chen Shuxing
(Institute of Mathematics, Fudan University, Shanghai 200433, China)

Dedicated to Professor Gu Chaohao on the occasion of his 70th birthday and his 50th year of educational work

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Abstract In this paper we discuss a Cauchy problem for nonlinear wave equation with delta initial data, including delta impulse and/or delta displacement. The solution of the Cauchy problem in appropriate sense is given. Meanwhile, the singularity structure of the solution is also described.

Key Words Nonlinear wave equation; singularity structure; delta initial data.
Classification 35L70, 35L67.

1. Introduction

Many physical phenomena involve propagation of waves, which are often described as the propagation of singularities of solutions to hyperbolic equations in mathematics. Therefore, singularity analysis for nonlinear hyperbolic equations attracted many mathematicians' attention (for instance, see [1] and the references cited there). Particularly, the essentially multi-dimensional phenomena caused by triple interaction of waves or by intersection of singularities of initial data are more interesting and challenging (See [2–6]). In previous works such a problem is mainly studied in H^s space with $s > n/2$. It means that the solution of the related problem exists in classical sense and it only carries weak singularities. However, the case when the solutions carry stronger singularity often appear in various physical problems, such as electromagnetic waves or seismic waves. As people knows, in this case the existence of solutions and the singularity analysis must be considered simultaneously. In [7] we considered a multi-dimensional Riemann problem for a nonlinear wave equation. That is a Cauchy problem with initial data being different constants in four quadrants. In this paper we will continue to

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study the case when the initial data have singularities as strong as delta measure, and the problem is still essentially multidimensional. We are going to consider two cases: delta impulse and delta displacement, both often appear in various physical problems. Here we refer readers to [8–10], where the propagation and interaction of delta waves are considered, particularly in one space dimensional case.

Since the solution of nonlinear equation will have singularity as strong as delta wave, the nonlinear composition may not work in usual sense. So the concept of the solution should be clarified first. As in [8], our solution will be understood as a sum of a regular part and a singular part, where the regular part is the solution of the equation with modified nonlinear part satisfying homogeneous initial data, and the singular part is the solution of corresponding linear equation with delta measure as initial data. Through careful analysis on the possible spreading of singularities of solution to 2-d wave equation we confirm the existence of the solution of our Cauchy problems. Comparing to the discussion in [8–10] we will more emphasize the singularity structure of the solution of nonlinear wave equation.

Our Cauchy problem for nonlinear wave equation is

$$\begin{cases} \Box u = f(u) \\ u\mid_{t=0} = \phi(x,y),\ u_t\mid_{t=0} = \psi(x,y) \end{cases} \tag{1.1}$$

where $\phi(x,y)$ and $\psi(x,y)$ has the form

$$\alpha_1\delta(x)H(y) + \alpha_2\delta(x)H(-y) + \beta_1\delta(y)H(x) + \beta_2\delta(y)H(-x)$$

or simply take the form $\delta(x)H(y)$, where $H(x)$ is Heaviside function. In this paper we always assume that $f(u)$ is smooth with respect to its argument in order to avoid the singularity caused by nonsmoothness of f. In the case of delta displacement we also assume that $f(u)$ and its derivative are bounded uniformly. For instance, $\Box u = \sin u$ is a model of such nonlinear equation.

Take $\{\delta_\varepsilon(x)\}$ as a normal δ-sequence, which converges $\delta(x)$ in distribution sense. For instance, $\delta_\varepsilon(x) = \dfrac{1}{\varepsilon}\alpha\Big(\dfrac{x}{\varepsilon}\Big)$, where $\alpha(x) \in C_c^\infty$ and $\int \alpha(s)\mathrm{d}s = 0$. For the convenience of our discussion, we also require $\operatorname{supp}\delta_\varepsilon \subset (-\varepsilon,\varepsilon)$ and $\delta_\varepsilon > 0$. Let $H_\varepsilon = H * \delta_\varepsilon$, $\widetilde{\delta}_\varepsilon(x,y) = \delta_\varepsilon(x)\delta_\varepsilon(y)$. Assume that $u_\varepsilon, v_\varepsilon$ and w_ε are the solutions of the following problems respectively:

$$\begin{cases} \Box u_\varepsilon = f(u_\varepsilon) \\ u_\varepsilon\mid_{t=0} = \phi * \widetilde{\delta}_\varepsilon,\ u_{\varepsilon t}\mid_{t=0} = \psi * \widetilde{\delta}_\varepsilon \end{cases} \tag{1.2}$$

$$\begin{cases} \Box v_\varepsilon = 0 \\ v_\varepsilon\mid_{t=0} = \phi * \widetilde{\delta}_\varepsilon,\ u_{\varepsilon t}\mid_{t=0} = \psi * \widetilde{\delta}_\varepsilon \end{cases} \tag{1.3}$$