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THE EXISTENCE OF GLOBAL SOLUTIONS TO A FLUID DYNAMIC MODEL FOR MATERIALS FOR KORTEWEG TYPE

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Dedicated to Professor Gu Chaohao on the occasion of his 70th birthday and his 50th year of educational work

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Abstract We discuss the existence of global solutions to a fluid dynamic model for materials of Korteweg type, and discuss the three dimensional nonisothermal motion. The higher order derivatives of density in the constitutive relation makes it necessary to introduce an interstitial working term in the energy equation. This makes the energy estimate more involved. Nevertheless, it is possible to show the existence of global solutions by the energy method.

Key Words Korteweg material; global solution.
Classification 35M10, 35Q35, 76N15, 35K30.

1. Introduction

In order to model the capillarity effect of materials, Korteweg [1] formulated a constitutive equation for the Cauchy stress that includes density gradients. In general, the constitutive relations having the gradients of density are incompatible with conventional thermodynamics. To remedy this difficulty, Dunn and Serrin [2] proposed the concept of interstitial working z having the form of $z = \mathbf{z} \cdot \mathbf{n}$ for every unit vector $\mathbf{n}$. Here, $\mathbf{z}$ is called the interstitial work flux representing spacial interactions of longer range. Employing this interstitial working into the balance of energy equation, they derived the following set of equations for the conservation of mass, the balance of linear momentum, the balance of energy, and the Clausius-Duhem inequality:

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1.1}$$

$$\rho \mathbf{u}_t + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla \cdot (\mathbf{T} + \mathbf{V}) \tag{1.2}$$

$$\rho \varepsilon_t + \rho \mathbf{u} \cdot \nabla \varepsilon = (\mathbf{T} + \mathbf{V}) \cdot \nabla \mathbf{u} - \nabla \cdot \mathbf{q} + \nabla \cdot \mathbf{z} \tag{1.3}$$

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$$\rho(\dot{\psi} + \eta\dot{\theta}) - (\mathbf{T} + \mathbf{V}) \cdot \nabla \mathbf{u} - \nabla \cdot \mathbf{z} + \frac{\mathbf{q} \cdot \nabla\theta}{\theta} \leq 0 \tag{1.4}$$

where $\dot{f} = f_t + \mathbf{u} \cdot \nabla f$ and

1. ρ is the density,
2. $\mathbf{u}$ is the velocity,
3. θ is the absolute temperature,
4. ε is the specific internal energy per unit mass,
5. η is the specific entropy per unit mass,
6. $\psi = \varepsilon - \theta\eta$ is the Helmholtz free energy,
7. $\mathbf{T}$ is the Cauchy stress tensor,
8. $\mathbf{V}$ is the viscosity tensor,
9. $\mathbf{q}$ is the heat flux vector.

For the elastic materials of Korteweg type, using the Clausius-Duhem inequality they have shown that the following forms of $\mathbf{z}$ and $\mathbf{T}$

$$\mathbf{z} = \rho\dot{\rho}\psi_{\mathrm{d}} \tag{1.5}$$

$$\mathbf{T} = (-\rho^2\psi_\rho + \rho\nabla \cdot (\rho\psi_{\mathrm{d}}))\mathbf{I} - \rho\mathbf{d} \otimes \psi_{\mathrm{d}} \tag{1.6}$$

are compatible with (1.12). Here, $\rho^2\psi_\rho(\rho, \theta, \mathbf{0}) = p(\rho, \theta)$ is the pressure and $\mathbf{d} = \nabla\rho$. They also have observed that the classical forms of viscosity and heat conductivity tensors are compatible. In what follows, we use the viscosity tensor and the heat flux vector given, respectively, by

$$\mathbf{V} = \mu\Big\{(\nabla\mathbf{u}) + (\nabla\mathbf{u})^T - \frac{2}{3}(\nabla \cdot \mathbf{u})\mathbf{I}\Big\}$$
$$\mathbf{q} = -\nabla\theta$$

For the Helmholtz free energy, we assume that it is given by

$$\psi(\rho, \theta, \mathbf{d}) = \sigma(\rho, \theta) + \lambda(\rho, \theta)(\mathbf{d} \cdot \mathbf{d}), \quad \lambda > 0$$

where σ and λ are smooth functions of their arguments. In this case

$$\eta = -\sigma_\theta - \lambda_\theta(\mathbf{d} \cdot \mathbf{d})$$
$$\varepsilon = \sigma - \theta\sigma_\theta + (\lambda - \theta\lambda_\theta)\mathbf{d} \cdot \mathbf{d}$$

As in the classical fluid, we assume that $e = \sigma - \theta\sigma_\theta$ and $S = -\sigma_\theta$ satisfies

$$de = \theta dS - pd\Big(\frac{1}{\rho}\Big)$$
$$p_\rho > 0, \quad e_\theta > 0, \quad \sigma_{\theta\theta} < 0 \tag{1.7}$$