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TOEPLITZ AND POSITIVE SEMIDEFINITE COMPLETION PROBLEM FOR CYCLE GRAPH*

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Abstract *We present a sufficient and necessary condition for a so-called C_n^k pattern to have positive semidefinite (PSD) completion. Since the graph of the C_n^k pattern is composed by some simple cycles, our results extend those given in [1] for a simple cycle. We also derive some results for a partial Toeplitz PSD matrix specifying the C_n^k pattern to have PSD completion and Toeplitz PSD completion.*

Key words *Partial matrix, Toeplitz matrix, completion problem, C_n^k pattern.*

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1 Introduction

A partial matrix is a matrix in which some entries are specified, while the remains entries are free to be chosen(from a certain set). A completion of a partial matrix is the conventional matrix resulting from a particular choice of values for the unspecified entries. For most of matrix completion problems there are some obvious conditions that must be satisfied for a completion of a certain class to exist. For example, for real symmetric positive semidefinite (PSD) completions, all completely specified submatrices must be symmetric PSD. A partial matrix which satisfies such a condition is called a partial PSD matrix. In this paper another class of matrices which we concern is Toeplitz matrix. It is known that an $n \times n$ matrix $A = (a_{ij})$ is called a symmetric Toeplitz matrix if $a_{i,j} = r_{|i-j|}$ for all $i, j = 1, 2, \ldots, n$. So a partial symmetric Toeplitz matrix is a partial symmetric matrix and if an entry in position (i, j) is specified then all entries in positions $(i + l, j + l)$ (mod n) are also specified and these (specified) entries are equal. Other

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types of partial matrices are defined similarly, see [4].

A pattern for $n \times n$ matrices is a list of positions of an $n \times n$ matrix, that is, a subset of $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$. A partial matrix specifies the pattern if its specified entries are exactly those listed in the pattern. A pattern Q is called symmetric if $(i, j) \in Q$ implies $(j, i) \in Q$. We assume throughout this paper that all the patterns we discuss are symmetric and include all diagonal positions, and when we define a pattern or calculate the number of the specified entries of a pattern, we shall ignore diagonal and symmetric positions of the pattern (e.g., Definition 1 below) and the entries in the positions. For a certain class of matrices, one important area of research is to decide for which patterns of specified entries all partial matrices in the class are completable to the class of matrices.

Among the types of matrix completion problems that have been studied are completion to positive definite (semidefinite) matrices [2],[7], to M-matrices and inverse M-matrices [5], to P-matrices and P_0 matrices[6],[4] and to contractions [8], etc.. For the positive definite completion problem, a solution was given by Grone, Johnson, Sa, and Wolkowicz in [2]. For the partial positive definite Toeplitz completion problem, an interesting open problem was presented in [7].

Now we give definition of the C_n^k pattern and describe the completion problems we shall consider.

Definition 1 Let $n/2 > k \geq 1$, the symmetric pattern

$$Q = \{(1, k+1), (2, k+2), \ldots, (n-k, n), (n-k+1, 1), \ldots, (n, k)\} \tag{1}$$

is called a C_n^k pattern. A Toeplitz C_n^k pattern is a C_n^k pattern with a restriction that the partial matrix specifying the pattern is a partial Toeplitz matrix.

Remark 1 In the definition, replacing k by $n-k$ gives the same pattern. So this is why we call it C_n^k pattern. It is trivial to show that when n is even and $k = n-k$, the pattern has PSD completion. So in the following, we always assume that $k < n-k$.

Remark 2 If we number the diagonals for an $n \times n$ symmetric Toeplitz matrix in the following way: The main diagonal is given the number 0 and then the diagonals are numbered in increasing order to the last, which becomes the $(n-1)$th, one. Then the (Toeplitz) C_n^k pattern is, in fact, the kth and $(n-k)$th diagonal.

In terms of the definition, our problems are as follows.

Problem 1 Whether does the C_n^k pattern have a PSD completion?

Without loss of generality, associated with a normalized of the data (see [1]), we assume