A DIRECT ALGORITHM FOR DISTINGUISHING NONSINGULAR $M$-MATRIX AND $H$-MATRIX*

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Abstract A direct algorithm is proposed by which one can distinguish whether a matrix is an $M$-matrix (or $H$-matrix) or not quickly and effectively. Numerical examples show that it is effective and convincible to distinguish $M$-matrix (or $H$-matrix) by using the algorithm.

Key words nonsingular $M$-matrix, nonsingular $H$-matrix, direct algorithm.

AMS(2000)subject classifications 15A48

1 Introduction

For many kinds of applications of $M$-matrices and $H$-matrices, the problem how to determine whether a matrix is an $M$-matrix (or $H$-matrix) or not arouses many researchers interesting. Recently, some iterative methods have been proposed for distinguishing $H$-matrices (see [2-5]). However, these methods have a common drawback, that is, it is not possible to determine the number of steps of iteration, and when $A$ is not an $H$-matrix, a wasteful computation is necessary. A direct algorithm has been proposed in [6], but it is only useful when matrices are symmetrical. In this paper, to conquer these drawbacks, we propose a new direct algorithm.

2 A direct algorithm for distinguishing $M$-matrix

Let $R^{n \times n}$ denote the set of all $n \times n$ real matrices. $A = (a_{ij}) \in R^{n \times n}$ is said to be an $M$-matrix if $a_{ij} \leq 0$, for $i \neq j$, and $A^{-1} \geq 0$.

Lemma 1[1] Let $A = (a_{ij}) \in R^{n \times n}$ be an $M$-matrix, then any principle submatrix of $A$ is an $M$-matrix.

Lemma 2[1] Let $A = (a_{ij}) \in R^{n \times n}$, its off-diagonal entries are all non-positive, then $A$ is

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* Foundation item: This work is supported by the Science Foundations of the Education Department of Yunnan Province (03Z169A) and the Science Foundatons of Yunnan University (2003Z013B). Received: Sep. 11, 2004.
an $M$-matrix if and only if successive principle minor of $A, D_K > 0, k = 1, \cdots, n$.

From Lemma 2, we can immediately obtain the following lemma.

**Lemma 3** Let $A = (a_{ij}) \in \mathbb{R}^{2 \times 2}$, and $a_{ij} \leq 0$, $i \neq j, a_{ii} > 0$, then $A$ is an $M$-matrix if and only if determinant of $A$, $\det A > 0$.

**Theorem 1** Let $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \in \mathbb{R}^{n \times n}$, where $B_{12} \leq 0, B_{21} \leq 0, B_{11}$ is a $2 \times 2$ square matrix and $B_{22}$ is an $(n-2) \times (n-2)$ square matrix, in which their diagonal entries are all positive and off-diagonal entries are all non-positive. Then $B$ is an $M$-matrix if and only if determinant of $B_{11} > 0$ and $B_{22} - B_{21}B_{11}^{-1}B_{12}$ is an $M$-matrix.

**Proof** Necessity: Suppose $B$ is an $M$-matrix, then

$$B^{-1} = \begin{bmatrix} B_{11}^{-1} + B_{11}^{-1}B_{12}(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1}B_{21}B_{11}^{-1} & -B_{11}^{-1}B_{12}(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1} \\ -(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1}B_{21}B_{11}^{-1} & (B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1} \end{bmatrix} \geq 0,$$

and $B_{11}$ and $B_{22}$ are $M$-matrices by Lemma 1. Hence, $\det B_{11} > 0$ by Lemma 3, and $B_{11}^{-1} \geq 0, (B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1} \geq 0$. For $B_{12} \leq 0, B_{21} \leq 0$, we have $B_{22}B_{11}^{-1}B_{12} \geq 0$, and off-diagonal entries of matrix $B_{22} - B_{21}B_{11}^{-1}B_{12}$ are all non-positive. So, $B_{22} - B_{21}B_{11}^{-1}B_{12}$ is an $M$-matrix.

Sufficiency: Suppose $\det B_{11} > 0$ and $B_{22} - B_{21}B_{11}^{-1}B_{12}$ is an $M$-matrix, then by Lemma 3, we have that $B_{11}$ is an $M$-matrix, so

$$(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1} \geq 0, \quad B_{11}^{-1} \geq 0.$$ 

Therefore

$$B_{11}^{-1} + B_{11}^{-1}B_{12}(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1}B_{21}B_{11}^{-1} \geq 0,$$

$$-(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1}B_{21}B_{11}^{-1} \geq 0,$$

$$-B_{11}^{-1}B_{12}(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1} \geq 0.$$

From these inequalities, we have

$$B^{-1} = \begin{bmatrix} B_{11}^{-1} + B_{11}^{-1}B_{12}(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1}B_{21}B_{11}^{-1} & -B_{11}^{-1}B_{12}(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1} \\ -(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1}B_{21}B_{11}^{-1} & (B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1} \end{bmatrix} \geq 0.$$

Thus $B$ is an $M$-matrix.

From Theorem 1, we propose the following algorithm A.

**Algorithm A**

**Input** The given matrix $B = (b_{ij}) \in \mathbb{R}^{n \times n}$.

**Step 1** Set $B = B^{(m)}$, and $m = 0$.

**Step 2** Partition $B^{(m)}$ into a $2 \times 2$ block matrix

$$B^{(m)} = \begin{bmatrix} B_{11}^{(m)} & B_{12}^{(m)} \\ B_{21}^{(m)} & B_{22}^{(m)} \end{bmatrix},$$