English Series | NUMERICAL MATHEMATICS A Journal of Chinese Universities | Vol.14, No.3 Aug. 2005

# FOURIER REGULARIZATION FOR DETERMINING SURFACE HEAT FLUX FROM INTERIOR OBSERVATION BASED ON A SIDEWAYS PARABOLIC EQUATION*

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**Abstract** *In this paper we consider a non-standard inverse heat conduction problem for determining surface heat flux from an interior observation which appears in some applied subjects. This problem is ill-posed in the sense that the solution (if it exists) does not depend continuously on the data. A Fourier method is applied to formulate a regularized approximation solution, and some sharp error estimates are also given.*

**Key words** *Inverse heat conduction; ill-posed problem; sideways parabolic equation; Fourier regularization; heat flux; error estimate.*

**AMS(2000)subject classifications** 47A52, 35R25, 35R30

## 1 Introduction and preliminary

In several engineering contexts, it is sometimes necessary to determine the surface temperature and heat flux in a body from a measured temperature history at a fixed location inside the body [1]. For the standard case, i.e., for the following sideways heat equation:

$$\begin{cases} u_t = u_{xx}, & x > 0, t > 0, \\ u(x,0) = 0, & x \geq 0, \\ u(1,t) = g(t), & t \geq 0, \quad u(x,t)|_{x\to\infty} \quad \text{bounded}, \end{cases} \tag{1}$$

the determination of surface temperature has been discussed by many authors by some different methods[2-6]. However, as it is said in [1]: "the heat flux is more difficult to calculate accurately

* Supported by the NNSF of China (No. 10271050), and the NSF of Gansu Province, China (ZS021-A25-001-Z)
Received: Sep. 1, 2004.

than the surface temperature" and the theoretical results are few [2], [7]. In this paper we consider the following non-standard inverse heat conduction problem in the quarter plane which appears in some applied subjects [8], [9]:

$$\begin{cases} u_t - u_x = u_{xx}, & x > 0, t > 0, \\ u(x,0) = 0, & x \geq 0, \\ u(1,t) = g(t). & t \geq 0, \quad u(x,t)|_{x\to\infty} \quad \text{bounded}, \end{cases} \tag{2}$$

and we now only pay attention to the determining of heat flux distribution on the interval $x \in [0,1)$.

As we consider problem (2) in $L^2(\mathbb{R})$ with respect to variable $t$, we extend $u(x,\cdot)$, $g(t) := u(1,t)$, $f(t) := u(0,t)$ and other functions appearing in the paper be zero for $t < 0$. As a solution of problem (2) we understand a function $u(x,t)$ satisfying (2) in the classical sense, and for every fixed $x \in [0,\infty)$, the temperature functions $u(x,\cdot)$ and heat flux $u_x(x,\cdot)$ belong to $L^2(\mathbb{R})$. We assume that there exists an a priori bound for $f(t) := u(0,t)$ :

$$\|f\|_p \leq E, \qquad \text{for some} \quad p \geq 0, \tag{3}$$

where

$$\|f\|_p := \left( \int_{-\infty}^{\infty} (1+\xi^2)^p |\hat{f}(\xi)|^2 \mathrm{d}\xi \right)^{\frac{1}{2}}, \tag{4}$$

and

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi t} f(t) \mathrm{d}t \tag{5}$$

is the Fourier transform of function $f(t)$. Let $g(t)$ and $g_\delta(t)$ be the exact and measured data at $x = 1$ of the solution $u(x,t)$ respectively, which satisfy

$$\|g_\delta - g\| \leq \delta, \tag{6}$$

where $\|\cdot\|$ denotes the norm of $L^2(\mathbb{R})$.

It is easy to see by taking the Fourier transform for variable $t$ in (2) that in the frequency domain the solution $u(x,t)$ of problem (2), if it exists, will satisfy the following problem:

$$\begin{cases} \hat{u}_{xx}(x,\xi) + \hat{u}_x(x,\xi) - i\xi\hat{u}(x,\xi) = 0, & x > 0, \xi \in \mathbb{R}, \\ \hat{u}(1,\xi) = \hat{g}(\xi), & \xi \in \mathbb{R}, \\ \hat{u}|_{x\to\infty}, & \text{bounded}. \end{cases} \tag{7}$$

The characteristic equation of the ordinary differential equation in (7) is

$$\lambda^2 + \lambda - i\xi = 0$$

and hence the roots of this equation are $\lambda = \dfrac{-1 \pm \sqrt{1+i4\xi}}{2}$, where $\sqrt{1+i4\xi}$ denotes the principal square root of $1+i4\xi$ and

$$\sqrt{1+i4\xi} = \sqrt[4]{1+16\xi^2} e^{i\frac{1}{2}\arg(1+i4\xi)}. \tag{8}$$