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Mortar Upwind Finite Volume Element Method with Crouzeix-Raviart Element for Parabolic Convection Diffusion Problems

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Abstract. In this paper, we study the semi-discrete mortar upwind finite volume element method with the Crouzeix-Raviart element for the parabolic convection diffusion problems. It is proved that the semi-discrete mortar upwind finite volume element approximations derived are convergent in the H^1- and L^2-norms.

Key words: Mortar upwind finite volume element method; Crouzeix-Raviart element; parabolic convection diffusion problems; error estimates.

AMS subject classifications: 65N55, 65N22, 65N30

1 Introduction

The mortar element method was first introduced by Bernardi, Maday and Patera in [2]. From then on, this method as a special nonconforming domain decomposition technique has aroused many researchers' attention because different types of discretizations can be employed in different parts of the computational domain. We refer to [2-5, 9, 10, 12, 18, 22] and the cited references there for details.

In the mortar element method, the computational domain is first decomposed into a polygonal partition. The meshes on different subdomains need not match across subdomain interfaces. The basic idea of this method is to replace the strong continuity condition on the interfaces between different subdomains by the so-called mortar condition. This condition guarantees the optimal discretization schemes, that is, the global discretization error is bounded by the sum of the optimal errors on different subdomains.

The finite volume element methods, also called the generalized difference methods in China, are popular in computational fluid mechanics due to their conservation properties of the original problems. In the past several decades, professors Li Ronghua *et al.* have systematically studied the finite volume element methods and obtained many important results. Interested readers are referred to the monographs [14, 15] for the general presentation of the finite volume element methods, and to [1, 6, 7, 11, 13, 16, 17, 19, 20, 23] and the references therein for details.

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Recently, Ewing, Lazarov, Lin and Lin [9] have considered mortar finite volume element approximations of the second-order self-adjoint elliptic problems. The discretization is based on the Petrov-Galerkin method with a solution space of continuous piecewise linear functions over each subdomain and a test space of piecewise constant functions. Bi and Li [3] have studied the mortar finite volume element method based on the mortar Crouzeix-Raviart finite element space and developed optimal order error estimates in the H^1- and L^2-norms.

In this paper, we construct and analyze the semi-discrete mortar upwind finite volume element method with the Crouzeix-Raviart element for parabolic convection diffusion problems. We use the mortar finite volume element method to discretize the diffusion term, and mortar upwind difference schemes to discretize the convection term, and establish error estimates in the H^1- and L^2-norms.

The remainder of this paper is organized as follows. In Section 2 we describe the parabolic convection diffusion problems, give the triangulation $\mathcal{T}_h$ of the computational domain Ω and the dual partition $\mathcal{T}_h^*$ of $\mathcal{T}_h$. Section 3 presents the semi-discrete mortar upwind finite volume element method for the parabolic convection diffusion problems. In Section 4, we get the error estimates in H^1- and L^2-norms.

In this paper, the notation of Sobolev spaces and associated norms and semi-norms are the same as those in Ciarlet [8], and C denotes the positive constant independent of the mesh parameter and the number of the subdomians, and may be different at different occurrences.

2 Notation and preliminaries

Consider the following parabolic convection diffusion problem on a bounded polygonal domain $\Omega \subset \mathcal{R}^2$:

$$\begin{cases} u_t - \nabla\cdot(A(x)\nabla u) + \nabla\cdot(b(x)u) = f, & x\in\Omega,\ \ 0<t\le T,\\ u(x,t)=0, & x\in\partial\Omega,\ \ 0<t\le T,\\ u(x,0)=u_0(x), & x\in\Omega. \end{cases} \tag{1}$$

We assume that $A = (a_{ij}(x))_{i,j=1}^2$ is a symmetric and uniformly positive definite matrix in Ω, $a_{ij} \in W^{1,\infty}(\overline{\Omega}),\ 1 \le i,j \le 2,\ b(x) \in (W^{1,\infty}(\overline{\Omega}))^2$. In this paper, in order to get the existence and uniqueness of the approximation solution in Section 3, we further assume that $\nabla\cdot b \ge 0$.

In this paper, we consider a geometrically conforming version of the mortar upwind finite volume element method, i.e., Ω is divided into non-overlapping polygonal subdomains Ω_i, $\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i$, with $\overline{\Omega}_i \cap \overline{\Omega}_j$ being an empty set or a vertex or an edge for $i \ne j$.

Each subdomain Ω_i is triangulated to produce a regular mesh $\mathcal{T}_h^i$ with the mesh parameter h_i, where h_i is the largest diameter of the elements in $\mathcal{T}_h^i$. The triangulations of subdomains generally do not align at the subdomain interfaces. Let Γ_{ij} denote the open straight line segment which is common to $\overline{\Omega}_i$ and $\overline{\Omega}_j$ and let Γ denote the union of all interfaces between the subdomains, i.e., $\Gamma = \cup\partial\Omega_i\backslash\partial\Omega$. We assume that the endpoints of each interface in Γ are vertices of $\mathcal{T}_h^i$ and $\mathcal{T}_h^j$. Let $\mathcal{T}_h$ denote the global mesh $\cup_i\mathcal{T}_h^i$ with $h = \max\limits_{1\le i\le N} h_i$.

Since the triangulations on two adjacent subdomains are independent, the interface $\overline{\Gamma}_{ij} = \overline{\Omega}_i\cap\overline{\Omega}_j$ is provided with two different and independent 1-D meshes, which are denoted by $\mathcal{T}_h^i(\Gamma_{ij})$ and $\mathcal{T}_h^j(\Gamma_{ij})$, respectively. We define one of the sides of Γ_{ij} as a mortar one, the other as a non-mortar one, denoted by γ_i and δ_j, respectively. Let $\Omega_{M(\Gamma_{ij})}$ denote the mortar domain of Γ_{ij} and $\Omega_{NM(\Gamma_{ij})}$ the non-mortar domain of Γ_{ij}. Define $u_{\gamma_i}^M$ and $u_{\delta_j}^{NM}$ to be the traces of $u|_{\Omega_{M(\Gamma_{ij})}}$ and $u|_{\Omega_{NM(\Gamma_{ij})}}$ on Γ_{ij}, respectively. Define CR nodal points as the midpoints of the edges of