NUMERICAL MATHEMATICS
A Journal of Chinese Universities (English Series)

Vol. **15** No. 1, pp. 50-59
February 2006

Numerical Quadratures for Hadamard Hypersingular Integrals †

Youjian Shen

Department of Mathematics, Hainan Normal University, Haikou 571158, China/Department of Mathematics, Zhongshan University, Guangzhou 510275, China.

Received October 13, 2003; Accepted (in revised version) August 23, 2004

Abstract. In this paper, we develop Gaussian quadrature formulas for the Hadamard finite part integrals. In our formulas, the classical orthogonal polynomials such as Legendre and Chebyshev polynomials are used to approximate the density function $f(x)$ so that the Gaussian quadrature formulas have degree $n-1$. The error estimates of the formulas are obtained. It is found from the numerical examples that the convergence rate and the accuracy of the approximation results are satisfactory. Moreover, the rate and the accuracy can be improved by choosing appropriate weight functions.

Key words: Gaussian quadrature; finite part; hypersingular integral; orthonormal polynomial.

AMS subject classifications: 65D30,65D32

1 Introduction

The numerical methods for the hypersingular integrals are most frequently encountered in many problems of mechanics. Particularly, they have been applied to solve the elasticity problems ([1-3]). In this paper,we consider hypersingular integrals of the form

$$I(t) = \text{f.p.} \int_a^b \frac{f(x)}{(x-t)^2} w(x) \mathrm{d}x, \quad t \in (a,b) \tag{1.1}$$

where f.p. denotes the finite part integral in the sense of Hadamard which is divergent in the classical sense, and $f(x)$ is a regular function on the interval $[a,b]$. The integral (1.1) is of second-order singularity. This type of integrals arises in the mechanics problems and the numerical methods of partial differential equations and integral equations, which have attracted considerable attention from researchers (see, for example, [1-8]). In numerical analysis of integrals arising from partial differential equations the chief difficulties in many cases are not only the loss

*Correspondence to: Youjian Shen, Department of Mathematics, Hainan Normal University, Haikou 571158, China/Department of Mathematics, Zhongshan University, Guangzhou 510275, China. Email: syjian@hainnu.edu.cn

†This research is supported in part by China NSF under grant 10071096, NSF of Guangdong under grant 990228, NSF of Hainan under grant 80525 and the One Hundred Distinguished Young Chinese Scientists Program of the Chinese Academy of Sciences from Yuesheng Xu.

of smoothness of the solution but also (and more crucially) the singularity of the solution. When smooth function are to be integrated , ordinary numerical methods are adequate, but when singular functions are to be integrated, the situation would be not reliable and satisfactory. Hence , it is desirable to have the simple and efficient quadrature formulas for the hypersingular integrals.

One of efficient methods for evaluating hypersingular integral (1.1) is generalizing the classical Gaussian quadrature rule. This was first done by Kutt[9] who has developed a set of Gaussian quadrature formulas. Ioakimidis, Pitta[10], Tsamasphyros and Dimou[11] developed the theorem of Kutt's. In the above works, the nodes and weights generally are complex numbers. This can reduce the precision in the numerical evaluation since $I(t)$ is real-valued. This is because the imaginary part will be required to cancel out exactly. Hui and Shia[12] have generalized Kutt's works to constant real nodes and weights that use classical orthogonal polynomials such as Legendre and Chebyshev, where the weight function is $w(x) = 1$ or $w(x) = \sqrt{1-x^2}$. Although Hui's method has more precision than that of Kutt's, it has some issues need to be further addressed: First the classical Gaussian quadrature formulas in general have degree $2n-1$, which integrate all polynomials up to its degree exactly. But Hui's formulas do not have any degree result. Second Hui's Gaussian quadrature formulas require parameter t not equal to the roots of orthogonal polynomials that will limit their application in the practical computation. The third Hui's method uses a deduction:

$$\text{if} \quad \int_{-1}^{1} \frac{f(x)}{x-t} w(x)\mathrm{d}x \approx g(t), \quad \text{then} \quad \frac{d}{dt}\int_{a}^{b} \frac{f(x)}{x-t} w(x)\mathrm{d}x \approx g'(t), \tag{$*$}$$

which is clearly coarse and would cause greater error and it is difficult to give an error estimate.

In this paper, we also use classical orthogonal polynomials such as Legendre and Chebyshev to establish Gaussian quadrature formulas for Cauchy principal value integrals and then use them to develop Gaussian quadrature formulas for finite part integrals . Nodes and weights in our approach are real-valued like Hui's. However, we avoid directly to use the formula ($*$), but we apply the polynomials (the classical orthogonal polynomials or Taylor expansions) to approximate the considered functions. Consequently, we can prove that our Gaussian formulas have degree $n-1$ and the formulas do not require the parameter t not equal to the roots of orthogonal polynomials that is more convenient in practical computations. Moreover the inaccurate deduction ($*$) is not used in our methods and we can establish error estimates for our Gaussian quadrature formulas and it seems possible that the accuracy of our formulas would be better than Hui's.

The paper is organized as follows. In Section 2, we develop Gaussian quadrature formulas for finite part integrals (1.1). In Section 3, the Gaussian quadrature formulas established in Section 2 are applied to the three concrete cases of weight function: $w(x) = 1, w(x) = \sqrt{1-x^2}$ and $w(x) = 1/\sqrt{1-x^2}$, and the error estimates of the formulas are given respectively. Finally, we present the results of the numerical experiments in Section 4.

2 Gaussian quadrature formulas

To simplify the exposition, we will take $a = -1, b = 1$ in (1.1), but our results are valid in the more general cases for finite a, b.

Definition 2.1. Suppose that $f(x)$ is a real function defined on $[-1, 1]$. If the limit

$$\lim_{\varepsilon\to 0}\left(\int_{-1}^{t-\varepsilon} \frac{f(x)}{(x-t)^2}\mathrm{d}x + \int_{t+\varepsilon}^{1} \frac{f(x)}{(x-t)^2}\mathrm{d}x - \frac{2f(x)}{\varepsilon}\right) \tag{2.1}$$