Approximation Theorems of Moore-Penrose Inverse by Outer Inverses†

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Abstract. Let $X$ and $Y$ be Hilbert spaces and $T$ a bounded linear operator from $X$ into $Y$ with a separable range. In this note, we prove, without assuming the closeness of the range of $T$, that the Moore-Penrose inverse $T^+$ of $T$ can be approximated by its bounded outer inverses $T^n$ with finite ranks.

Key words: Moore-Penrose inverse; outer inverse.

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1 Introduction and preliminaries

Let $X$ and $Y$ be two Hilbert spaces and $T$ a bounded linear operator from $X$ into $Y$. We use $D(T)$, $N(T)$ and $R(T)$, respectively, to denote the domain, null space and range of $T$.

Recall that a linear operator $T^\# : Y \mapsto X$ is said to be an outer inverse of $T$ if $T^\# T T^\# = T^\#$. A linear operator $T^+ : Y \mapsto X$ is said to be the Moore-Penrose inverse of $T$ [1], if $T^+$ satisfies $D(T^+) = R(T) \oplus R(T)^\perp$ and the four Moore-Penrose equations:

$$TT^+ T = T, \quad T^+ T T^+ = T^+ \text{ on } D(T^+),$$
$$T^+ T = I - P_{N(T)}, \quad TT^+ = P_{R(T)} \text{ on } D(T^+),$$

where $P_{(\cdot)}$ is the orthogonal projection onto the subset in the parenthesis.

It is well known that the approximation theory of Moore-Penrose inverse of linear operators plays an important role in various areas of nonlinear analysis and optimization. The approximations of the Moore-Penrose inverse have been studied in the literature such as [1-7]. For an operator with closed range, Z. Ma and J. Ma gave an approximation theorem of the Moore-Penrose inverse by outer inverses with finite ranks [5]. A natural question is whether the Moore-Penrose inverse of an operator with non-closed range can be approximated by its bounded outer inverses.

A fundamental distinction between the case of an operator with closed range and the case of an...
operator with non-closed range is that the Moore-Penrose inverse of an operator with non-closed range turns out to be an unbounded operator. Therefore, approximations to such a generalized inverse by bounded operators can converge only in the point-wise sense at best. In this paper, without assuming the closeness of $R(T)$, we give an approximation theorem which asserts that the Moore-Penrose inverse of an operator with separable range can be approximated by its bounded outer inverses with finite ranks. Moreover, because of the stability of the bounded outer inverse [8], our theorems are very useful in computing the Moore-Penrose inverse and in finding the least-square solution of the operator equation.

2 Main results

Theorem 2.1. Let $X$ and $Y$ be Hilbert spaces and $T$ a bounded linear operator from $X$ into $Y$ with a separable range. For each positive integer $n$, there exists a bounded outer inverse $T^+_n$ of $T$ with finite rank $n$ such that

$$D(T^+) = \left\{ y : \lim_{n \to \infty} T^+_n y \text{ exists} \right\},$$

and if $y \in D(T^+)$, then

$$T^+ y = \lim_{n \to \infty} T^+_n y.$$

Proof. Without loss of generality, we suppose that $R(T)$ is infinite dimensional. Choose a sequence $Y_1 \subset Y_2 \subset \cdots \subset Y_n \subset \cdots$ of finite dimensional subspaces of $\overline{R(T)} \subset Y$ with $\dim Y_n = n$ and $\bigcup_{n=1}^\infty Y_n = \overline{R(T)} = N(T^*)^\perp$, where $T^*$ is the adjoint operator of $T$. Let $X_n = T^* Y_n$. Then

$$X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots \subset R(T^* Y_n) = N(T)^\perp,$$

$$\dim X_n = n$$

and

$$\bigcup_{n=1}^\infty X_n = \overline{R(T^*)}.$$

Indeed,

$$\overline{R(T^*)} = \overline{R(T^* T)} = \overline{T^* (R(T))} = \overline{T^* R(T)} = \overline{T^* (\bigcup_{n=1}^\infty Y_n)}$$

$$= \overline{T^* (\bigcup_{n=1}^\infty Y_n)} = \bigcup_{n=1}^\infty T^* Y_n = \bigcup_{n=1}^\infty X_n.$$

Let $P_n$ and $Q_n$ denote the orthogonal projectors from $Y$ onto $Y_n$ and from $X$ onto $X_n$ respectively. Put

$$T_n = P_n T.$$

Then $T_n$ is a bounded linear operator with closed range. Also, $N(T_n)^\perp = R(T^*_n) = R(T^* P_n) = X_n$ and $R(T_n) = Y_n$, since $R(T_n)^\perp = N(T^*_n) = N(T^* P_n) = N(P_n) = Y_n^\perp$. In order to construct an outer inverse of $T$, we define $T^+_n \in B(Y, X)$ as follows:

$$T^+_n y = \begin{cases} (T_n|_{X_n})^{-1} y, & y \in Y_n, \\ 0, & y \in Y_n^\perp, \end{cases}$$

Thus $T^+_n$ is a bounded outer inverse of $T$ with $\dim R(T^+_n) = \dim X_n = n$. In fact, obviously,

$$T^+_n y = T^+_n P_n y \quad \text{for all} \quad y \in Y.$$