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A Simple Analytical and Numerical Approach for Pricing Compound Options†

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Abstract. A compound option is simply an option on an option. In this short paper, by using a martingale technique, we obtain an analytical formula for pricing compound European call options. Numerical results are given to explain some economic phenomenon.

Key words: Compound option; European call option; Brownian motion; Girsanov theorem; bisection method.

AMS subject classifications: 60H30, 74S30, 91B28

1 Introduction

A compound option is defined to be an option on an option. Therefore, a compound option has two expiration dates and two strike prices. For instance, we consider an European style call on a call. On the first expiration date T, the option holder has the right to buy a new call using the strike price K_T. The new call will have an expiration date U and a strike price K_U. An important paper [4] on the valuation of compound option was written by Geske in 1979. Up to now, many works on compound options have been published, see $[1, 3, 9, 11]$ and references therein.

In fact, the pricing of many other derivatives can be modeled as compound options. For example, by visualizing the underlying stock as an option on the firm value, an option on stock of a firm that expires earlier than the maturity date of the debt issued by the firm can be regarded as a compound option on the firm value [4]. On the expiration of the option, i.e., the first expiration date of the compound option, the holder chooses to acquire the stock or not. The decision depends on whether the stock as a call on the firm value is more valuable than the strike price. Also, the strategy of sale before completion of development of a house is another typical example of compound options [8].

In this paper, following a technique proposed in [1], we give an analytical formula for pricing compound European call options. Numerical results are given to explain some economic phenomenon. We remark that although we only consider the case of a call on a call option, the

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generalization to some other type of compound options, e.g., a call on a put, a put on a call, etc, is straightforward.

We give a mathematical framework of compound options first. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions [3,6,10], and let $W(t)$ be a standard $(\{\mathcal{F}_t\}, \mathbf{P})$-Brownian motion. We consider a financial market consists of an underlying asset whose price $S(t)$ is given by the following stochastic differential equation:

$$\mathrm{d}S(t) = \mu(t)S(t)\mathrm{d}t + \sigma(t)S(t)\mathrm{d}W(t),$$

with $S(0) = s_0 > 0$, and a money market account whose price $R(t)$ is given by

$$\mathrm{d}R(t) = \gamma(t)R(t)\mathrm{d}t, \quad R(0) = 1. \tag{1}$$

Here $\mu(t)$ and $\sigma(t)$ are $\{\mathcal{F}_t\}$-adapted processes such that for any $t > 0$,

$$\mathbf{E}\big[\textstyle\int_0^t \mu(s)\mathrm{d}s\big] < \infty \quad \text{and} \quad \mathbf{E}\big[\textstyle\int_0^t \sigma^2(s)\mathrm{d}s\big] < \infty,$$

and $\gamma(t)$ is a bounded and measurable deterministic function.

Let $U > 0$ be a fixed time. For an European type call option on $S(\cdot)$ with maturity U and strike price K_U, it is well-known that its claim X is given by

$$X = \big(S(U) - K_U\big)^+.$$

Assume that there exists an $\{\mathcal{F}_t\}$-adapted process $\lambda(t)$ which satisfies

$$\lambda(t) \cdot \sigma(t) = \mu(t) - \gamma(t), \quad 0 \le t \le U,$$

and the Novikov's condition [1]:

$$\mathbf{E}\left[\exp\left\{\int_0^U \frac{1}{2}\lambda^2(s)\mathrm{d}s\right\}\right] < \infty.$$

Then by the Girsanov's theorem [1,7], we can define a new equivalent probability measure $\tilde{\mathbf{P}}$ on $(\Omega, \mathcal{F}_U)$ by

$$\mathrm{d}\tilde{\mathbf{P}}/\mathrm{d}\mathbf{P} = \exp\Big\{-\int_0^U \lambda(s)\mathrm{d}W(s) - \frac{1}{2}\int_0^U \lambda^2(s)\mathrm{d}s\Big\}$$

such that $\tilde{W}(t) = W(t) + \int_0^t \lambda(s)\mathrm{d}s$, $0 \le t \le U$, is a standard $(\{\mathcal{F}_t\}, \tilde{\mathbf{P}})$-Brownian motion, and the price process $S(t)$ satisfies

$$\mathrm{d}S(t) = \gamma(t)S(t)\mathrm{d}t + \sigma(t)S(t)\mathrm{d}\tilde{W}(t), \quad 0 \le t \le U, \tag{2}$$

i.e., $\tilde{\mathbf{P}}$ is the risk-neutral probability measure (the equivalent martingale measure). Hence, by the well-known fact in mathematical finance [1], we know that the value of X for the numeréraire $R(t)$ in (1) at time $0 \le t \le U$ is given by

$$X(t) = e^{-\int_t^U \gamma(s)\mathrm{d}s}\mathbf{E}_{\tilde{\mathbf{P}}}\big[\big(S(U) - K_U\big)1_{\{S(U)>K_U\}} \mid \mathcal{F}_t\big],$$

where $\mathbf{E}_{\tilde{\mathbf{P}}}[\cdot]$ denotes the expected value with respect to the probability $\tilde{\mathbf{P}}$. Now, we can introduce a compound call option Y on $X(t)$ with maturity $0 < T < U$ and strike price K_T, i.e.,

$$Y(T) = \big(X(T) - K_T\big)^+.$$

We emphasize here that for compound options, to our knowledge, only the constant case of μ, σ and γ are considered in the literature, see [1].