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Partition of Unity for a Class of Nonlinear Parabolic Equation on Overlapping Non-Matching Grids†

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Abstract. A class of nonlinear parabolic equation on a polygonal domain $\Omega \subset \mathbb{R}^2$ is investigated in this paper. We introduce a finite element method on overlapping non-matching grids for the nonlinear parabolic equation based on the partition of unity method. We give the construction and convergence analysis for the semi-discrete and the fully discrete finite element methods. Moreover, we prove that the error of the discrete variational problem has good approximation properties. Our results are valid for any spatial dimensions. A numerical example to illustrate the theoretical results is also given.

Key words: Nonlinear parabolic equation; finite element method; overlapping non-matching grids; partition of unity.

AMS subject classifications: 65F10, 65N30, 65N15

1 Introduction

Since Huang and Xu [1] proposed a finite element method for overlapping non-matching grids based on partition of unity, the new finite element method has been attracting many authors' interest. Recently, there have been some studies of applying the finite element method to overlapping grids. These studies are within the framework of mortar finite elements or Lagrange multipliers [4-6]. The partition of unity method that has its roots in Babǔska and Melenk in [2,3], has been used for the numerical solutions of the parabolic problems [7-9]. Both linear elliptic and parabolic problems are studied [1,11]. However, the discrete case of the nonlinear parabolic problem has not been investigated when overlapping grids and non-matching grids are involved. In this paper, following the ideas of Huang and Xu, we propose a finite element method by introducing a conforming finite element space and by using an argument of the partition of unity type for a class of nonlinear parabolic problem.

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The rest of this paper is organized as follows. In Section 2, we give a brief description for the continuous nonlinear parabolic problem and the discretization of overlapping sub-domains. We also construct a globally conforming finite element space based on partition of unity. In Section 3, we give a few examples of the partition of unity function. We give the main results of the paper in Sections 4 and 5. They include the convergence analysis of the semi-discrete finite element solution based on partition of unity and the fully discrete finite element solution for the nonlinear parabolic problem. In Section 6, a numerical example is presented.

2 Construction of a global conforming subspace using the partition of unity

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with smooth boundary $\partial\Omega$, Γ be a closed subset of $\partial\Omega$. By $H_0^1(\Omega;\Gamma)$, we denote the closure in $H^1 - topology$ of $C^\infty(\overline{\Omega})$ functions that vanish in a neighborhood of Γ. Consider the following initial-boundary value problem for a class of nonlinear parabolic differential equation:

$$\begin{cases} \partial_t u - \nabla\cdot(a(u)\nabla u) = f(u), & \text{for } x\in\Omega, t\in(0,T], \\ u(x;t) = 0, & \text{for } x\in\partial\Omega, t\in(0,T], \\ u(x;0) = g(x), & \text{for } x\in\Omega, \end{cases} \tag{1}$$

where a and f are smooth functions defined on $\mathbb{R}$ such that

$$0<\mu\le a(u)\le M, \quad |a'(u)|+|f'(u)|\le B, \quad \text{for } u\in\mathbb{R}. \tag{2}$$

Assume that the above problem admits a unique solution which is smooth enough for our purposes.

Now we begin our discussion of overlapping grids. We consider an overlapping domain decomposition of Ω, namely, we take $\Omega_1, \Omega_2, ..., \Omega_s$ to be overlapping sub-domains satisfying

$$\Omega = \bigcup_{i=1}^{s} \Omega_i.$$

We assume that each Ω_i is partitioned by a quasi-uniform finite element triangulation (or quadrilateral) J^{h_i} of maximal mesh size h_i, which are different from each other. Assume d_i is the minimal overlapping size of Ω_i with its neighboring sub-domains. Denote

$$J^h = \bigcup_{i=1}^{s} J^{h_i}, \quad h = \max_{1\le i\le s}\{h_i\}, \quad d = \min_{1\le i\le s}\{d_i\}.$$

We shall use the notation $\lesssim$ and $\gtrsim$, i.e., when we write $x_1 \lesssim y_1, x_2 \gtrsim y_2$, we mean that there exist constants c_1, c_2, such that

$$x_1 \le c_1 y_1, \quad x_2 \ge c_2 y_2,$$

where c_i $(i=1,2)$ are constants independent of mesh size h.

For every sub-domain Ω_i and partition $J^{h_i} (i=1,2,...,s)$, we have the corresponding stationary finite element space:

$$V^{h_i}(\Omega_i) = \{v\in H_0^1(\Omega_i;\partial\Omega\cap\partial\Omega_i); v|_e \in P_{m_i+r-1}, e\in J^{h_i}, m_i\ge 1, r\ge 1\} \subset H^1(\Omega),$$