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Comparison Results Between Preconditioned Jacobi and the *AOR* Iterative Method

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Abstract

The large scale linear systems with M-matrices often appear in a wide variety of areas of physical, fluid dynamics and economic sciences. It is reported in [1] that the convergence rate of the IMGS method, with the preconditioner $I+S_\alpha$, is superior to that of the basic SOR iterative method for the M-matrix. This paper considers the preconditioned Jacobi (PJ) method with the preconditioner $P=I+S_\alpha+S_\beta$, and proves theoretically that the convergence rate of the PJ method is better than that of the basic AOR method. Numerical examples are provided to illustrate the main results obtained.

Keywords: Preconditioner; Jacobi iteration; AOR iteration; M-matrix; Linear system.
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1. Introduction

Consider the linear system

$$Ax=b, \tag{1.1}$$

where A is an $n\times n$ square matrix, x and b are n-dimensional vectors. To accelerate the convergence of the iteration method solving the linear system (1.1), the preconditioned methods are often used. In general, the preconditioned system of (1.1) is

$$PAx=Pb, \tag{1.2}$$

where the nonsingular matrix P is called the preconditioner. The systems (1.2) with the different preconditioners P were discussed by many authors (see, e.g., [1-3, 5, 7]).

In [2], the author considered the preconditioner $P=I+S_\alpha$, where

$$S_\alpha=(s_{ij})_{n\times n}=\begin{cases} -\alpha_i a_{i,j}, & j=i+1, \\ 0, & \text{otherwise}, \end{cases} \qquad 0\le\alpha_i\le 1. \tag{1.3}$$

Many authors considered this preconditioner (see, e.g., [1, 2, 5, 7]). In particular, it is proved in [1] that if A is a nonsingular M-matrix and the parameter satisfied $0<\omega\le 1$,

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then the asymptotic convergence rate of the IMGS method is faster than that of the SOR method and the Gauss-Seidel method.

In this paper, for the preconditioner $P = I + S_\alpha + S_\beta$ proposed in [3], where S_α is defined as above and

$$S_\beta = (s_{ij})_{n\times n} = \begin{cases} -\beta_i a_{i,j}, & j = i-1, \\ 0, & \text{otherwise}, \end{cases} \qquad 0 \le \beta_i \le 1, \tag{1.4}$$

we prove theoretically that if A is a nonsingular M-matrix and the entries of A and relaxation parameters of *AOR* method satisfy the P-condition defined in Section 3, then the asymptotic convergence rate of the PJ method is faster than that of the *AOR* method.

The rest of the paper is organized as follows. In Section 2, we give some concepts and some lemmas. In Section 3, we prove that the PJ method is superior to the *AOR* method under the P-condition. In Section 4, we give two numerical examples to illustrate the obtained results in Section 3.

2. Preliminaries

A square matrix $A = (a_{ij})_{n\times n}$ is called (nonsingular) M-matrix if $A = sI - B, B \ge 0$ and $(s > \rho(B))s \ge \rho(B)$, where $\rho(B)$ denotes the spectral radius of B.

For a nonsingular M-matrix A, without loss of generality, we always assume henceforward that

$$A = I - L - U, \tag{1.5}$$

where I is an identity matrix, $-L$ and $-U$ are strictly lower and upper triangular matrices obtained from A, respectively. Thus the iterative matrices of the classical Jacobi method and *AOR* method are

$$T_J = L + U$$

and

$$L_{r,\omega} = (I - rL)^{-1}[(1-\omega)I + (\omega - r)L + \omega U]$$

with the two parameters ω, r satisfying $0 \le r \le \omega \le 1$, respectively.

$A = M - N$ is said to be M-splitting if M is a nonsingular M-matrix and $N \ge 0$. A splitting $A = M - N$ is said to be weak regular, if $M^{-1} \ge 0$ and $M^{-1}N \ge 0$, $NM^{-1} \ge 0$. Obviously M-splitting is weak regular.

Lemma 2.1. [5] *Let A be irreducible, $A = M - N$ is an M-splitting. Then there is a positive vector x such that $M^{-1}Nx = \rho(M^{-1}N)x$.*

Lemma 2.2. [4] *Let $A = M - N$ be a weak regular splitting. Then $\rho(M^{-1}N) < 1$, if and only if $A^{-1} \ge 0$*

Lemma 2.3. [6] *Let T be a nonnegative matrix, x be a nonnegative nonzero vector and α a positive scalar. The following results hold.*

1. *If $Tx \ge \alpha x$, then $\rho(T) \ge \alpha$. Moreover, if $Tx > \alpha x$, then $\rho(T) > \alpha$.*