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A Quasi-Interpolation Satisfying Quadratic Polynomial Reproduction with Radial Basis Functions

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Abstract

In this paper, a new quasi-interpolation with radial basis functions which satisfies quadratic polynomial reproduction is constructed on the infinite set of equally spaced data. A new basis function is constructed by making convolution integral with a constructed spline and a given radial basis function. In particular, for twicely differentiable function the proposed method provides better approximation and also takes care of derivatives approximation.

Keywords: Radial basis functions; quasi-interpolation; equally spaced data; spline.
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1. Introduction

Buhmann [2] discussed 'quasi-interpolation' with radial basis functions whose form is

$$s_h(x) = \sum_{j \in Z} f(jh)\Psi(\frac{x}{h} - j), \quad x \in \mathbb{R}, \tag{1.1}$$

where f is the approximand and Ψ is a finite linear combination of radial basis functions

$$\Psi(x) = \sum_{|k| \leqslant N} \lambda_k \phi(|x - k|), \quad x \in \mathbb{R}, \tag{1.2}$$

where ϕ is a radial basis function. It is understood that s_h depends on h as well as f and $\Psi(x)$ is completely independent of f. The function s_h does not need to satisfy any interpolation property but it should be such that $s_h \approx f$ is good uniform approximation on real line by virtue of properties of the function $\Psi(x)$. The above form is particularly simple because f enters directly into the expression without any preprocessing, which is different from radial basis functions interpolation that need to solve a very large system of equations for obtaining coefficients.

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In order to guarantee the convergence of (1.1) we require the absolute sum

$$\sum_{j\in Z}\Psi(x-j) \tag{1.3}$$

to be uniformly bounded and absolutely convergent for all $x\in\mathbb{R}$ and by demanding

$$\sum_{j\in Z}\Psi(x-j)\equiv 1,$$

Buhmann chose multiquadric function $\sqrt{r^2+c^2}$ as radial basis function Φ and let Ψ be a second divided difference of ϕ

$$\Psi(x)=\frac{1}{2}\phi(|x-1|)-\phi(|x|)+\frac{1}{2}\phi(|x+1|). \tag{1.4}$$

In particular. The Ψ satisfies linear polynomial reproduction property, namely

$$\sum_{j\in Z}(a+bj)\Psi(x-j)=a+bx,\quad \text{for any}\quad x,a,b\in\mathbb{R}.$$

In convergence results, Buhmann gave an easy but quite representative convergence theorem.

Theorem 1.1. *Let f be twice differentiable and such that $\|f'\|_\infty$ and $\|f''\|_\infty$ are finite. Then for any nonnegative c in $\sqrt{r^2+c^2}$*

$$\|f-s_h\|_\infty=\mathcal{O}(h^2+c^2h^2|\log h|),\qquad h\to 0.$$

For the first derivatives, a similar convergence statement can be made

$$\|f'-s_h'\|_\infty=\mathcal{O}(h+c^2/h),\qquad h\to 0.$$

It is evident that the derivatives approximation is not satisfactory because it is necessary that c tends to zero in order for derivatives convergence. In fact, there is closely relation between the convergence and polynomial recovery because smooth functions can be locally approximated by Taylor polynomials which may be recovered by the 'Quasi-interpolation'. Therefore, it is necessary for high convergence order that polynomial of high orders need to be recovered by approximant.

In this paper, we construct a new 'quasi-interpolation' form with radial basis functions that satisfies quadratic polynomial reproduction property and obtain better convergence properties. In Section 2, we give the detailed process of the construction. The convergence analysis and some open problems are presented in Sections 3 and 4.