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Super Implicit Multistep Collocation Methods for Weakly Singular Volterra Integral Equations

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Abstract. In this paper, we consider super implicit multistep collocation methods for solving weakly singular Volterra integral equations. After the integral equations are regularized by smoothing transformations, super implicit multistep collocation methods are constructed. Convergence and the linear stability of these methods are analyzed. The theoretical result is also demonstrated by numerical examples.

AMS subject classifications: 65L20

Key words: Collocation method, Volterra integral equation, weakly singular kernel, convergence.

1. Introduction

Inspired by the promising numerical methods in [2, 10, 14, 23, 24], we present a class of super implicit multistep collocation methods (SIMCMs) for the linear weakly singular Volterra Integral Equations (VIEs)

$$u(x) = f(x) + \int_0^x K(x,y)u(y)dy, \quad x \in [0,X], \tag{1.1}$$

with the initial value $u(0) = u_0$. Here, $K(x,y) = (x-y)^{-\alpha}k(x,y)$, $f : [0,X] \to \mathbb{R}$ and $k : \Upsilon \to \mathbb{R}$ with $\Upsilon := \{(x,y) : 0 \le y \le x \le X\}$. Such equations have received a considerable amount of attention both numerically and theoretically (see, e.g., [6, 13, 18, 25] and the references therein).

Several authors, including Abdalkhani [1] and te Riele [21], were interested in one-step collocation approximations for (1.1). If the initial intervals of uniform partition are employed, the theoretical accuracy falls to less than 1, regardless of how we choose its degree. To preserve the optimal orders of convergence, graded meshes are applied and the resulting collocation methods for weakly singular VIEs exhibit the orders identical to the degrees of the collocation polynomials (see [4, 7]). However, a problem may

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arise with the use of strongly non-uniform grids for special subintervals of very small length near the singular point of the exact solution. This approach can cause significant roundoff errors in the calculations and leads to unstable behavior of numerical results. To avoid loss of precision, [12, 13, 15] used collocation methods on uniform or mildly graded meshes by means of smoothing transformations. The convergence properties of the methods are improved by introducing new suitable independent variables.

Meanwhile, the idea of the two-step and multistep collocation methods (MCMs) for solving VIEs was first developed by Conte et al. [9, 10]. Performance of these methods lies in the fact that they increase the convergence order without increasing the computational cost, except the cost due to the starting procedure. Recently, Fazeli, Hojjati and Shahmorad (see [14]) introduced SIMCMs which depend on fixed number of previous time steps and fixed number of collocation points in the currnet and next subintervals. In this paper, we use variable transformations to change the weakly singular equation (1.1) into a new VIE with better regularity. The new VIE is still weakly singular, but its solution is smooth enough (see [3]). After that, we construct SIMCMs for weakly singular VIEs, analyze the convergence and stability of the proposed methods.

The paper is organized as follows. In Section 2, the smoothing transformation is defined and the construction of SIMCMs for weakly singular VIEs is described. In Sections 3, the error bounds are obtained for these methods. In Section 4, the linear stability is analyzed for these methods. In Section 5, a brief description is given for the implementation of SIMCMs. Finally, some numerical experiments are given to show the convergence order of the proposed method.

2. Construction of SIMCMs for weakly singular VIEs

In this section, we describe the construction of SIMCMs for weakly singular VIEs (1.1). Before introducing the suitable transformations, we set the following weight function for convenience in expression

$$D_q(x) = \begin{cases} 1 & \text{for } q < 0, \\ (1 + |\log x|)^{-1} & \text{for } q = 0, \\ x^q & \text{for } q > 0, \end{cases}$$

where $x \in (0, X]$. We need also two sets of smooth functions as follows.

Definition 2.1. ([12]) *For $m \in \mathbb{N}$ and $\alpha < 1$, we denote $C^{m,\alpha}[0, X]$ as the set of all functions $u : [0, X] \to \mathbb{R}$ which are m times continuously differentiable in $(0, X]$ and satisfy*

$$\left| u^{(j)}(x) \right| \leq C D^{-1}_{\alpha+j-1}(x), \quad j = 1, \cdots, m.$$

Here, $C = C(u)$ is a positive constant for all $x \in (0, X]$.