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A Linearized Second-Order Difference Scheme for the Nonlinear Time-Fractional Fourth-Order Reaction-Diffusion Equation

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Abstract. This paper presents a second-order linearized finite difference scheme for the nonlinear time-fractional fourth-order reaction-diffusion equation. The temporal Caputo derivative is approximated by $L2\text{-}1_\sigma$ formula with the approximation order of $\mathcal{O}(\tau^{3-\alpha})$. The unconditional stability and convergence of the proposed scheme are proved by the discrete energy method. The scheme can achieve the global second-order numerical accuracy both in space and time. Three numerical examples are given to verify the numerical accuracy and efficiency of the difference scheme.

AMS subject classifications: 65M06, 65M12, 65M15
Key words: Fractional differential equation, Caputo derivative, high order equation, nonlinear, linearized, difference scheme, convergence, stability.

1. Introduction

Fractional differential equations (FDEs) [1] have attracted more and more attention because of their frequent appearance in various application in science and engineering. They are more appropriate for the description of memorial and hereditary properties of various materials and processes than the classical integer-order differential equations. A great deal of effort has been spent on attempting to find the stable methods for solving the FDEs. However, most FDEs can not get the exact solution. So various kinds of numerical methods have been used to seek the approximation solution for the FDEs, such as finite element method [2, 3], spectral method [4, 5], Parareal algorithm [6], finite difference method [7–28], and so on.

Finite difference method for solving fractional differential equations have been widely proposed and analyzed by many scholars. Some related works are introduced below.

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A shifted Grüwald-Letnikov (GL) formula with the approximation order of $\mathcal{O}(h)$ was proposed by Meerschaert in [7] to approximate the Riemann-Liouville derivative. Based on the shifted GL formula, Yuste et al. [8, 9] constructed the explicit and weighted averaged finite difference schemes. The stability of these schemes has been analyzed by the Von Neumann method. In [10], the authors constructed the difference schemes based on the GL formula for fractional sub-diffusion equation and reaction sub-diffusion equation, respectively. The stability and convergence of the difference schemes are proved by the Fourier method. The convergence order of the difference scheme is $\mathcal{O}(\tau+h^2)$. Cui et al. [11] constructed a compact ADI difference scheme with operator splitting method for two-dimensional time fractional diffusion equation based on the GL formula. The scheme is unconditionally stable.

To improve the accuracy of the GL formula, a weighted and shifted Grünwald difference (WSGD) operator in [12] was developed for the Riemann-Liouville derivative which can achieve second-order accuracy. Consequently, based on the WSGD formula, a second-order difference scheme for the time fractional diffusion-wave equation in [13] was derived based on WSGD formula. In [14], a second-order finite difference scheme was presented for solving two-dimensional space-fractional diffusion equations based on the WSGD formula. In [15], a linearized difference scheme is proposed for nonlinear Stokes' first problem for a heated generalized second grade fluid with fractional derivative. The time fractional derivative is approximated by WSGD formula. The numerical method is unconditionally stable with the global convergence order of $\mathcal{O}(\tau^2+h^4)$ in maximum norm.

There are also some different second-order accuracy numerical methods to deal with the fractional derivative. In [16], based on the Lubich method, a second-order difference approximation for the Riemann-Liouville fractional derivative by the GL formula with the generating function $(3/2-2z+z^2/2)^\alpha$ was derived. The scheme is convergent with the order of $\mathcal{O}(\tau^2+h^4)$. Du et al. [17] constructed some high-order difference schemes for the distributed-order time-fractional equations in both one and two space dimensions. Based on the composite Simpson formula and Lubich second-order operator, a difference scheme was presented with convergence order of $\mathcal{O}(\tau^2+h^4+\sigma^4)$. Sousa [18] proposed a second-order accuracy scheme to approximate the spatial fractional Caputo derivative via spline method.

These methods above can achieve the second-order accuracy for approximating the fractional derivative. However, they have very rigorous conditions on the initial value or the boundary value. The function $u(x,t)$ needs to be zero extended to be defined outside the domain considered and the extended function should have a certain smoothness at the initial value or the boundary.

Another frequently-used method based on the idea of the piecewise linear interpolation, called $L1$ formula, is applied to approximate the Caputo fractional derivative. Sun and Wu [19] presented the discrete difference schemes based on the $L1$ formula for the fractional diffusion-wave equation and sub-diffusion equation. They proved the unconditional stability and convergence of the difference schemes with the convergence order of $\mathcal{O}(\tau^{3-\alpha}+h^2)$ for the diffusion-wave equation and $\mathcal{O}(\tau^{2-\alpha}+h^2)$ for the