Numer. Math. Theor. Meth. Appl. Vol. **12**, No. 4, pp. 1213-1230
doi: 10.4208/nmtma.OA-2018-0122 November 2019

One-Step Multi-Derivative Methods for Backward Stochastic Differential Equations

Chengjian Zhang[1,2], Jingwen Wu[1,2] and Weidong Zhao[3,*]

[1] *School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, Hubei, China*
[2] *Hubei Key Laboratory of Engineering Modeling and Scientific Computing, Huazhong University of Science and Technology, Wuhan 430074, Hubei, China*
[3] *School of Mathematics, Shandong University, Jinan 250100, Shandong, China*

Received 29 November 2018; Accepted (in revised version) 21 April 2019

Abstract. This paper deals with numerical solutions of backward stochastic differential equations (BSDEs). For solving BSDEs, a class of third-order one-step multi-derivative methods are derived. Several numerical examples are presented to illustrate the computational effectiveness and high-order accuracy of the methods. To show the advantage of the methods, a comparison with θ-methods is also given.

AMS subject classifications: 60H35, 65C20, 60H10

Key words: Backward stochastic differential equations, one-step multi-derivative methods, θ-method, Itô-Taylor expansion, third-order accuracy.

1. Introduction

Consider the one-dimension backward stochastic differential equations (BSDEs) in a filtered complete probability space $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{0\le t\le T})$:

$$Y_t=\varphi(X_T)+\int_t^T f(s,Y_s)ds-\int_t^T Z_s dW_s, \qquad t\in[0,T], \tag{1.1}$$

where $f:\Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}$ is the generator of (1.1) with the Lipschitz condition and linear increment condition w.r.t. y, $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{0\le t\le T})$ a filtered complete probability space with $\{\mathcal{F}_t\}_{0\le t\le T}$ being the natural filtration of the one-dimensional Brownian motion $\{W_t\}_{0\le t\le T}$, $X_t=x_0+W_t$ with an assigned constant x_0, $\varphi:\mathbb{R}\to\mathbb{R}$ a given function with the linear increment condition and pair $(Y_t,Z_t):[0,T]\times\Omega\to\mathbb{R}\times\mathbb{R}$ an L^2-adapted solution of (1.1), i.e., it is $\mathcal{F}_t$-adapted and square integrable. In order to perform the subsequent analysis to BSDEs (1.1), we introduce the following notations:

*Corresponding author. *Email addresses:* `cjzhang@mail.hust.edu.cn` (C. J. Zhang), `ccwu1088@sina.com` (J. W. Wu), `wdzhao@sdu.edu.cn` (W. D. Zhao)

- $C^{l,k}([0,T]\times\mathbb{R})$: space of $\psi:(t,x)\in[0,T]\times\mathbb{R}\to\mathbb{R}$ with continuous partial derivatives up to l w.r.t. t and up to k w.r.t. x;
- $C_b^{l,k}([0,T]\times\mathbb{R})$: space of $\psi:(t,x)\in[0,T]\times\mathbb{R}\to\mathbb{R}$ with uniformly bounded partial derivatives up to l w.r.t. t and up to k w.r.t. x;
- $C^k(\mathbb{R},\mathbb{R})$: space of k times continuously differentiable function $f:\mathbb{R}\to\mathbb{R}$;
- $L^2_{\mathbb{F}}([0,T];\mathbb{R})$: space of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted processes X_t $(t\in[0,T])$ with $\mathbb{E}[\int_0^T|X_t|^2]<\infty$ a.s.

The unique solvability condition of (1.1) has been derived in Pardoux and Peng [10]. Moreover, Peng [11] presented a relationship between BSDEs (1.1) and a class of specific partial differential equations. These conclusions show that (1.1) has a unique solution $(Y_t,Z_t)\in L^2_{\mathcal{F}_t}([0,T];\mathbb{R})$ with

$$Y_t=u(t,W_t),\qquad Z_t=\nabla_x u(t,W_t),\qquad t\in[0,T],\tag{1.2}$$

where $\nabla_x u$ is the gradient of u w.r.t. x and $u(t,x)\in C_b^{1,2}([0,T]\times\mathbb{R})$ denotes the solution of the following Cauchy problem:

$$\begin{cases}\dfrac{\partial u}{\partial t}+\dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x)+f(t,u(t,x))=0, & t\in[0,T),\quad x\in\mathbb{R},\\ u(T,x)=\varphi(x), & x\in\mathbb{R}.\end{cases}\tag{1.3}$$

BSDEs (1.1) are a class of important equations and have the wide applications in financial mathematics, nonlinear martingale theory, game theory, stochastic control and so forth (cf. [2,12–14,16]). Nevertheless, it is hard to obtain the analytical solution of a BSDE. Thus, developing various numerical methods to solve BSDEs becomes a very significant topic. Up to now, a number of effective numerical methods for BSDEs have been presented, such as Crank-Nicolson methods [18,21], θ-method [1,22,23], Monte-Carlo methods [9,24], Runge-Kutta methods [4], spectral methods [6] and multistep methods [19,25–28].

Among the existed numerical methods for BSDEs (1.1), except some of the multistep methods and Runge-Kutta methods, no one-step method that the order exceeds two has been obtained up to now. In view of this, in the present paper, we consider the construction of a class of third-order one-step methods. The rest of the paper is organized as follows. In Section 2, for the subsequent analysis, some elementary concepts and results are reviewed. In Section 3, with the Itô-Taylor expansion and nonlinear Feynman-Kac formula, a class of one-step multi-derivative methods are derived, whose order can arrive at three. In Section 4, several numerical examples are presented to illustrate the computational effectiveness and numerical accuracy of the methods. Finally, in Section 5, a comparison between one-step multi-derivative methods and θ-method is given, which shows that the former has better advantage than the latter in computational accuracy.