

Auxiliary Equations Approach for the Stochastic Unsteady Navier–Stokes Equations with Additive Random Noise

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Received 2 April 2019; Accepted (in revised version) 11 August 2019

Abstract. This paper presents a Martingale regularization method for the stochastic Navier–Stokes equations with additive noise. The original system is split into two equivalent parts, the linear stochastic Stokes equations with Martingale solution and the stochastic modified Navier–Stokes equations with relatively-higher regularities. Meanwhile, a fractional Laplace operator is introduced to regularize the noise term. The stability and convergence of numerical scheme for the pathwise modified Navier–Stokes equations are proved. The comparisons of non-regularized and regularized noises for the Navier–Stokes system are numerically presented to further demonstrate the efficiency of our numerical scheme.

AMS subject classifications: 35R60, 65Mxx, 76Dxx

Key words: Stochastic Navier–Stokes equations, Martingale regularization method, Galerkin finite element method, white noise.

1. Introduction

In this paper, a Martingale regularization method is proposed to improve the computational efficiency and accuracy of the stochastic Navier–Stokes equations (SNSEs) with additive noise,

$$\left\{ \begin{array}{ll} \mathbf{d}\mathbf{u} - \nu \Delta \mathbf{u} \, dt + (\mathbf{u} \cdot \nabla) \mathbf{u} \, dt + \nabla p \, dt = \mathbf{f} \, dt + \sigma(t) \Delta_\theta \, d\mathbf{W} & \text{in } (0, T] \times D \times \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } [0, T] \times D \times \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } [0, T] \times \partial D \times \Omega, \\ \mathbf{u} = \mathbf{u}_0 & \text{on } D \times \Omega, \end{array} \right. \quad (1.1)$$

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where $[0, T]$ denotes a time interval of interest; $D \in \mathbb{R}^d$, $d = 2, 3$ a space domain with Lipschitz boundary; Ω a sample space of all the possible outcomes; ν the viscosity parameter; $\mathbf{u} = (u_1, \dots, u_d) : [0, T] \times D \times \Omega \rightarrow \mathbb{R}^d$ the velocity field; $p : [0, T] \times D \times \Omega \rightarrow \mathbb{R}$ the corresponding pressure field; $\mathbf{f} : [0, T] \times D \times \Omega \rightarrow \mathbb{R}^d$ a random forcing term; $\sigma(t) : [0, T] \rightarrow \mathbb{R}$ a scalar function; and $\mathbf{u}_0 : D \times \Omega \rightarrow \mathbb{R}^d$ an initial condition. The notation $\Delta_\theta := (-\Delta)^{-\theta}$ denotes a fractional Laplace operator with $\theta \in \mathbb{R}^+$ [16] and $\mathbf{W} = (W_1, \dots, W_d) : [0, T] \times D \times \Omega \rightarrow \mathbb{R}^d$ denotes an infinite-dimensional Wiener process. Further, assume that W_l , $l = 1, \dots, d$ are Wiener processes with bounded, self-adjoint, positive semidefinite covariance operator Q which has eigenvalues $\{\gamma_l > 0\}$ and eigenfunctions $\{e_l(x)\}$. Then from [3], the Wiener process \mathbf{W} can be characterized by

$$\mathbf{W}(t, x, \omega) = \sum_{l=1}^{\infty} \gamma_l^{1/2} e_l(x) \beta_l(t, \omega), \quad (1.2)$$

where $\beta_l(t, \omega) = (\beta_1^l, \dots, \beta_d^l)$ is a sequence of real-valued independent and identically distributed (*i.i.d.*) standard Brownian motions. In this paper, the trace $\text{tr } Q = \sum_{l=1}^{\infty} \gamma_l$ is not required to be finite. The infinite case can be incorporated with some feasible regularization parameter θ of the fractional Laplace operator.

The crucial part of the Martingale regularization method for (1.1) is having in hand previous knowledge of an auxiliary stochastic process associated with the stochastic Stokes equations

$$\begin{cases} d\boldsymbol{\eta} - \nu \Delta \boldsymbol{\eta} dt + \nabla \zeta dt = \sigma(t) \Delta_\theta d\mathbf{W} & \text{in } (0, T] \times D \times \Omega, \\ \nabla \cdot \boldsymbol{\eta} = 0 & \text{in } [0, T] \times D \times \Omega, \\ \boldsymbol{\eta} = \mathbf{0} & \text{on } [0, T] \times \partial D \times \Omega, \\ \boldsymbol{\eta} = \boldsymbol{\eta}_0 & \text{on } D \times \Omega, \end{cases} \quad (1.3)$$

where $\boldsymbol{\eta} : [0, T] \times D \times \Omega \rightarrow \mathbb{R}^d$ is the corresponding auxiliary velocity and $\zeta : [0, T] \times D \times \Omega \rightarrow \mathbb{R}$ is the corresponding pressure. Theoretically, the stochastic Stokes equations (1.3) are somehow relevant to the divergence free projection of one stochastic parabolic equations. Moreover, the solution of Eqs. (1.3) with time-space white noise in two dimension or higher has less regularities, e.g., $\boldsymbol{\eta}(t) \notin L^2_{\mathbb{P}}(\Omega, \mathbf{H}^1(D))$ [3]. The fractional Laplace operator Δ_θ is given here to regularize the noise term $d\mathbf{W}$ so as to make the pathwise solution of (1.3) smoother and more amenable to computation. Therefore, system (1.1) with a cylindrical Wiener process or a greatly non-smoother Wiener process can be covered. Setting $\mathbf{u} = \boldsymbol{\xi} + \boldsymbol{\eta}$, $p = \zeta + \pi$, the induced velocity $\boldsymbol{\xi}$ and pressure π satisfy the modified Navier–Stokes equations

$$\begin{cases} d\boldsymbol{\xi} - \nu \Delta \boldsymbol{\xi} dt + (\boldsymbol{\xi} + \boldsymbol{\eta}) \cdot \nabla (\boldsymbol{\xi} + \boldsymbol{\eta}) dt + \nabla \pi dt = \mathbf{f} dt & \text{in } (0, T] \times D \times \Omega, \\ \nabla \cdot \boldsymbol{\xi} = 0 & \text{in } [0, T] \times D \times \Omega, \\ \boldsymbol{\xi} = \mathbf{0} & \text{on } [0, T] \times \partial D \times \Omega, \\ \boldsymbol{\xi} = \boldsymbol{\xi}_0 & \text{on } D \times \Omega. \end{cases} \quad (1.4)$$