

# The Generalized Order Tensor Complementarity Problems

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Received 16 November 2018; Accepted (in revised version) 11 March 2019

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**Abstract.** The main propose of this paper is devoted to study the solvability of the generalized order tensor complementarity problem. We define two problems: the generalized order tensor complementarity problem and the vertical tensor complementarity problem and show that the former is equivalent to the latter. Using the degree theory, we present a comprehensive analysis of existence, uniqueness and stability of the solution set of a given generalized order tensor complementarity problem.

**AMS subject classifications:** 15A18, 15A69, 65F15, 65F10

**Key words:** Generalized order tensor complementarity problems, tensor complementarity problems, structured tensors, degree theory.

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## 1. Introduction

Over the past decade, the research of finite-dimensional variational inequality and complementarity problems [9, 19, 24, 36] has been rapidly developed in the theory of existence, uniqueness and sensitivity of solutions, theory of algorithms and the application of these techniques to transportation planning, regional science, socio-economic analysis, energy modeling and game theory. The tensor complementarity problem (TCP), which is a natural generalization of the linear complementarity problem (LCP) and a special case of the nonlinear complementarity problem (NCP), is a new topic emerged from the tensor community, inspired by the growing research on structured tensors. Huang and Qi [26] reformulated the multilinear game (a class of  $N$ -person noncooperative games) as the TCP and showed that finding a Nash equilibrium point of

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the multilinear game is equivalent to finding a solution of the resulted TCP. The readers can be recommended [2, 6, 21, 26, 35, 41, 42] for a thorough survey of the existence of the solution set of the TCP.

Recently, some researchers focus on numerical algorithms for solving the TCPs and the interested readers can be recommended, e.g., [15, 16, 23, 26, 33, 47]. Che et al. [7] considered the stochastic tensor complementarity problem via the theory of stochastic  $R_0$  tensors. Barbagallo et al. [3] studied some variational inequalities on a class of structured tensors. Wang et al. [46] introduced the tensor variational inequality, where the involved function is the sum of an arbitrary given vector and a homogeneous polynomial defined by a tensor. The interested readers can be referred to [27–29] for the basic theory, solution methods and applications of tensor complementarity problems.

Gowda and Sznajder [22] studied the solution set of the generalized order linear complementarity problem (GOLCP) and showed that this problem is equivalent to the generalized linear complementarity problem (henceforth called the vertical linear complementarity problem, abbreviated by VLCP, as in [11]) considered by Cottle and Dantzig [10], which established basic existence results via Lemke’s algorithms. The readers can be recommended [1, 13, 17, 25, 30, 37, 45] and their references for a survey of the GOLCP. Gowda [20] introduced the concept of degree of an  $\mathbb{R}_0$  tensor and showed that the degree of an  $\mathbb{R}_0$  tensor is one.

In this paper, we study a special case of the generalized order complementarity problem, named as the generalized order tensor complementarity problem (GOTCP). GOTCPs can be viewed as the generalizations of GOLCPs and TCPs. In order to study the solution set of the GOTCP, we define the degree of any arbitrary tensor and the classes of tensors, based on the degree of an  $\mathbb{R}_0$  tensor. We give a comprehensive analysis of existence, uniqueness and stability issue connected with the GOTCP. As we see, the degree theory approach allows us to prove existence results directly and under relaxed assumptions. Degree theory plays an important role in stability results as well.

It is a brief description of various sections. In Section 2, we define the GOTCP the vertical tensor complementarity problem (VTCP) and introduce the classes of  $\mathbb{R}_0$  tensors,  $\mathbb{P}$  tensors,  $\mathbb{Q}$  tensors and so on, which can be used to study the solution set of the TCP. In Section 3, we introduce block tensors of various types and prove existence results in a general setting. In Section 4 we obtain the necessary and sufficient conditions for the zero vector being the unique solution of any given GOTCP. In Section 5, we give the sufficient conditions for the solution set and an isolated solution of the GOTCPs being stable. In Section 6, we conclude this paper and present some issues which will be considered in the future.

## 2. Preliminaries

Higher-order equivalents of vectors (first order) and matrices (second order) are called *higher-order tensors*, *multi-dimensional matrices*, or *multi-way arrays*. A tensor is an  $N$ -order array with dimension  $I$  of numbers denoted by script notation  $\mathcal{A} \in$