Numer. Math. Theor. Meth. Appl. Vol. **13**, No. 1, pp. 99-130
doi: 10.4208/nmtma.OA-2019-0053 February 2020

A Well-Conditioned, Nonconforming Nitsche's Extended Finite Element Method for Elliptic Interface Problems

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Received 25 March 2019; Accepted (in revised version) 6 July 2019

Abstract. In this paper, we introduce a nonconforming Nitsche's extended finite element method (NXFEM) for elliptic interface problems on unfitted triangulation elements. The solution on each side of the interface is separately expanded in the standard nonconforming piecewise linear polynomials with the edge averages as degrees of freedom. The jump conditions on the interface and the discontinuities on the cut edges (the segment of edges cut by the interface) are weakly enforced by the Nitsche's approach. In the method, the harmonic weighted fluxes are used and the extra stabilization terms on the interface edges and cut edges are added to guarantee the stability and the well conditioning. We prove that the convergence order of the errors in energy and L^2 norms are optimal. Moreover, the errors are independent of the position of the interface relative to the mesh and the ratio of the discontinuous coefficients. Furthermore, we prove that the condition number of the system matrix is independent of the interface position. Numerical examples are given to confirm the theoretical results.

AMS subject classifications: 65N12, 65N15, 65N30

Key words: Elliptic interface problems, NXFEM, nonconforming finite element, condition number.

1. Introduction

In this paper we consider the following elliptic interface problem:

$$\begin{cases} -\nabla\cdot(a(x)\nabla u)=f & \text{in}\quad \Omega_1\cup\Omega_2,\\ [u]=g_D,\quad [(a(x)\nabla u)\cdot\mathbf{n}]=g_N & \text{on}\quad \Gamma,\\ u=0 & \text{on}\quad \partial\Omega, \end{cases} \tag{1.1}$$

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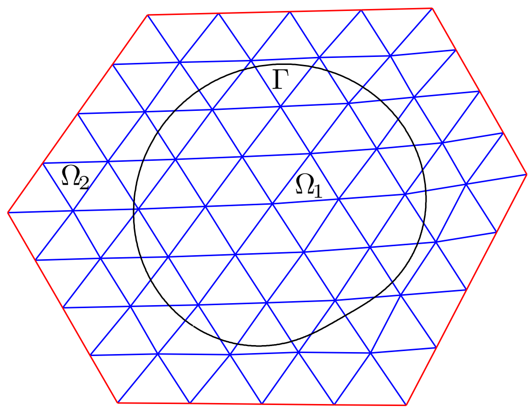

Figure 1: A sample domain Ω and an unfitted mesh.

where $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ is a bounded and convex domain in $\mathbb{R}^2$, Ω_1 and Ω_2 are two subdomains of Ω separated by the interface $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ (see Fig. 1 for an illustration). $[v] = v|_{\Omega_1} - v|_{\Omega_2}$ denotes the jump of v across the interface Γ, and $\mathbf{n}$ is the unit normal vector to Γ pointing from Ω_1 to Ω_2. We assume that $a(x) = a_i$ for $x \in \Omega_i$ with constants $a_i > 0$, $i = 1, 2$, and denote by $a_{\min} = \min_{i=1,2}\{a_i\}$, $a_{\max} = \max_{i=1,2}\{a_i\}$. We further assume that the interface Γ is C^2-smooth.

The problem (1.1) is often occurred in material sciences and fluid dynamics which involves two or more distinct materials or fluids with different densities, conductivities or permeabilities. Much attention has been paid to the numerical methods for this problem in recent decades. We refer to the immersed boundary element method [1], the finite difference methods (see, for example, the immersed interface method [2] and the ghost fluid method [3]) and the finite element methods (see, for example, the multiscale finite element method [4], the immersed finite element method [5–10], the ghost fluid method [44], the splitting collocation method [43], the weak Galerkin finite element method [42, 45], the unfitted finite element method [11–16, 18] and the mortar element method [19]). In this paper, we focus on the numerical methods related to the finite element implementations.

Since the global regularity of the solution is low due to the nature of the interface and discontinuity of the coefficients in the equation, the performance of the standard finite element method is not very well unless the interface coincides with mesh lines. One strategy to solve the interface problems with accurate approximation is the interface-fitted grid methods where the discontinuity is directly captured by the mesh (see [19–24] and the references therein). However, it is difficult and time consuming to generate a body fitted grid for the interface problems with the complicated interface. In particular, for the moving interface problems, such a difficulty is more severe because of the expensive remeshing at each time step to maintain a good mesh. Therefore, various unfitted grid methods for the problem (1.1) have been proposed in the literature, where the interface can be arbitrarily located with respect to the mesh