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A Stabilized Low Order Finite Element Method for Three Dimensional Elasticity Problems

Gwanghyun Jo[1] and Do Y. Kwak[2,*]

[1] *Department of Mathematics, Kunsan National University, Gunsan Si, Jeollabuk-do, Republic of Korea*
[2] *Department of Mathematical Sciences, KAIST, Daejeon, Republic of Korea*

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Abstract. We introduce a low order finite element method for three dimensional elasticity problems. We extend Kouhia-Stenberg element [12] by using two nonconforming components and one conforming component, adding stabilizing terms on the associated bilinear form to ensure the discrete Korn's inequality. Using the second Strang's lemma, we show that our scheme has optimal convergence rates in L^2 and piecewise H^1-norms even when Poisson ratio ν approaches $1/2$. Even though some efforts have been made to design a low order method for three dimensional problems in [11, 16], their method uses some higher degree basis functions. Our scheme is the first true low order method. We provide three numerical examples which support our analysis. We compute two examples having analytic solutions. We observe the optimal L^2 and H^1 errors for many different choice of Poisson ratios including the nearly incompressible cases. In the last example, we simulate the driven cavity problem. Our scheme shows non-locking phenomena for the driven cavity problems also.

AMS subject classifications: 65N12, 65N30
Key words: Elasticity equation, low order finite element, Kouhia-Stenberg element, locking free, Korn's inequality.

1. Introduction

We consider the following type of elasticity equation in a convex polyhedral domain Ω in $\mathbb{R}^3$:

$$-\operatorname{div}\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{f} \qquad \text{in} \quad \Omega, \tag{1.1a}$$

$$\mathbf{u} = 0 \qquad \text{in} \quad \partial\Omega, \tag{1.1b}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement variable and $\mathbf{f} \in [L^2(\Omega)]^3$ is an external force. We may also consider the pure traction problems, but we choose the Dirichlet

*Corresponding author. *Email addresses:* `gwanghyun@kunsan.ac.kr` (G. Jo), `kdy@kaist.ac.kr` (D. Y. Kwak)

boundary conditions just for simplicity of presentation. Here, the strain tensor $\boldsymbol{\epsilon}(\mathbf{u})$ and the stress tensor $\boldsymbol{\sigma}(\mathbf{u})$ are as usual,

$$\boldsymbol{\epsilon}_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad \boldsymbol{\sigma}(\mathbf{u}) = 2\mu\boldsymbol{\epsilon}(\mathbf{u}) + \lambda \mathrm{tr}(\boldsymbol{\epsilon}(\mathbf{u}))\mathbf{I},$$

where $\mathbf{I}$ is 3×3 by identity matrix. The Lamé constants μ and λ are given in terms of modulus of elasticity $E > 0$ and Poisson's ratio $0 < \nu < 1/2$,

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}.$$

We note that as $\nu \to 1/2$, the parameter λ goes to infinity, as in incompressible case.

It is well known that conforming low order finite elements for solving elasticity problem usually yields a locking phenomena as the Poisson ratio approaches to 1/2 [15]. For nonconforming elements, the associated bilinear form fails to satisfy the discrete Korn's inequality [9]. Hence the coerciveness does not hold. The analyses regarding the locking phenomena in [9, 15] are developed in two dimensional problems. However, by restricting to R^2, we can see that low order methods are also locking in R^3. Thus, to obtain optimal convergence rates using Lagrangian type of finite element methods, one must use polynomial of order ≥ 4, when the material is nearly incompressible [15]. However, some nonconforming elements of degree ≥ 2 converges uniformly as the Poisson ratio approaches $1/2$ [9].

Some efforts have been made to avoid locking phenomena using lower order nonconforming methods. One often uses reduced integration or macro element technique [4, 9]. Some people apply the mixed methods [5] to elasticity equations (see [14]). Other approaches are to design the finite element (FE) space or to modify the bilinear form to satisfy the discrete Korn's inequality. Kouhia-Stenberg (KS) [12] used conforming-nonconforming pair for two dimensional problems, while Hansbo, et al. [10] used nonconforming pair with stability terms to enforce coerciveness. However, it was shown [11, 16] that a straightforward extension of KS element to three dimensional case is impossible. For example, the pair (P_n^1, P_n^1, P_c^1) would not satisfy the Korn's inequality if we restrict it to the first two components. The authors in [16] used Q_2-conforming space in one of the components while the authors in [11] used bubble functions of degree 3 in one of the components.

In this paper, we present a simple extension of KS element to three dimensional elasticity problems using the pair (P_n^1, P_n^1, P_c^1). Instead we add stability terms on the first two component, which yields the smaller number of degrees of freedom than the spaces introduced in [11, 16]. The concept of adding stabilizing term on the bilinear form is motivated by [10, 13]. With the aid of the stabilizing term, we were able to prove that our scheme is stable, i.e., the bilinear form is coercive with respect to broken H^1-norm. In this way, we obtain a new extension of KS method to 3D element using only piecewise linear functions, while the number of unknowns is about 69 percent of (P_n^1, P_n^1, P_n^1) elements (see Example 4.2). We provide optimal error estimates in