Numer. Math. Theor. Meth. Appl. Vol. **13**, No. 2, pp. 320-333
doi: 10.4208/nmtma.OA-2019-0077 May 2020

# An Algorithm That Localizes and Counts the Zeros of a $C^2$-Function

Norbert Hungerbühler and Rui Wu*

*Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland*

Received 23 May 2019; Accepted (in revised version) 9 September 2019

**Abstract.** We describe an algorithm that localizes the zeros of a given real $C^2$-function $f$ on an interval $[a,b]$. The algorithm generates a sequence of subintervals which contain a single zero of $f$. In particular, the exact number of zeros of $f$ on $[a,b]$ can be determined in this way. Apart from $f$, the only additional input of the algorithm is an upper and a lower bound for $f''$. We also show how the intervals determined by the algorithm can be further refined until they are contained in the basin of attraction of the Newton method for the corresponding zero.

**AMS subject classifications**: 30C15

**Key words**: Number of zeros on an interval.

## 1. Introduction

A fundamental task, both in pure and in numerical analysis, is to localize and to count the zeros of a given function $f$ in a certain region. For example the Argument Principle and Rouché's theorem for holomorphic functions allow to determine the number of zeros of $f$, counted with their multiplicity, in a bounded domain of $\mathbb{C}$ with sufficiently regular boundary (see, e.g., [5] for an overview of methods used for analytic functions). In real analysis Sturm's theorem, a refinement of Descartes' Sign Rule and the Fourier-Budan theorem, allows to count the exact number of roots of a polynomial $p$ with simple roots on a real interval (see, e.g., [3, 9, 10]). Several attempts have been made to transfer the method of Sturm to trigonometric polynomials: see [1, 7, 8]. The mentioned methods are restricted to holomorphic functions and polynomials, respectively. If we consider a function $f$ which is merely continuous, the theorem of Bolzano yields the information that $f$ has at least one zero on an interval $[a,b]$ if $f$ has opposite signs at its endpoints. An iterative method to enclose all real zeros of a real function in an interval of minimal length has been discussed in [6]. Only recently, a method which counts the zeros of a function $f$ under only mild regularity conditions on an interval

*Corresponding author. *Email address:* norbert.hungerbuehler@math.ethz.ch (N. Hungerbühler)

 

$[a, b]$ has been proposed in [4]. In particular, if an $L^\infty$-bound on the second derivarive of

$$\frac{f'^2 - ff''}{f^2 + f'^2}$$

is known, then the number of zeros of $f$ can be computed by evaluating $f, f'$ and $f''$ on a fine enough grid. In the present paper, we would like to present an algorithm which works for functions $f \in C^2([a, b])$ and which only requires an upper and a lower bound for $f''$ (see Section 3). The termination proof will be given in Section 4. This algorithm can in particular be used as a preconditioner for other algorithms like Newton's or Aitken's method to compute the zeros numerically (see Section 5). But first we need a refinement of Bolzano's theorem.

## 2. Refinement of Bolzano's theorem

We start from the following elementary observation:

**Lemma 2.1.** *Let $f, g \in C^2([a, b])$ be functions with $f(a) = g(a), f(b) = g(b)$ and $f'' \leq g''$ on $[a, b]$. Then, we have $f \geq g$ on $[a, b]$.*

*Proof.* Let $h := g - f$. Then we have $h(a) = h(b) = 0$ and $h'' \geq 0$ on $[a, b]$. Hence, $h$ is convex and therefore we conclude $h \leq 0$ on $[a, b]$. □

Now we want to formulate a criterion which ensures that a function $f \in C^2([a, b])$ does *not* have a zero. To do so, we will use Lemma 2.1 with a comparison function $g$ of constant second derivative. First, we note the following:

**Lemma 2.2.** *Let $g : [a, b] \to \mathbb{R}$ be the unique function $g$ with $g(a) = \alpha$, $g(b) = \beta$ and $g'' = \gamma$ on $[a, b]$. Then we have*

$$\min g = \begin{cases} \dfrac{1}{2}\left(\alpha + \beta - \dfrac{\gamma}{4}(b-a)^2 - \dfrac{(\beta-\alpha)^2}{\gamma(b-a)^2}\right), & \text{if } \gamma(b-a)^2 > 2|\beta - \alpha|, \\ \min\{\alpha, \beta\}, & \text{otherwise.} \end{cases} \tag{2.1}$$

*Proof.* The proof is an elementary calculation. □

As a consequence of Lemma 2.2 we now get:

**Corollary 2.1.** *Let $f \in C^2([a, b])$ be a function with $f(a) = \alpha > 0$, $f(b) = \beta > 0$ and $f'' \leq \gamma$ on $[a, b]$. If*

$$\gamma(b-a)^2 \leq 2|\beta - \alpha| \qquad \text{or} \qquad 4(\alpha + \beta) > \gamma(b-a)^2 + \frac{4}{\gamma}\left(\frac{\beta - \alpha}{b - a}\right)^2,$$

*then $f$ has no zero on $[a, b]$.*