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A Dynamical Method for Solving the Obstacle Problem

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Abstract. In this paper, we consider the unilateral obstacle problem, trying to find the numerical solution and coincidence set. We construct an equivalent format of the original problem and propose a method with a second-order in time dissipative system for solving the equivalent format. Several numerical examples are given to illustrate the effectiveness and stability of the proposed algorithm. Convergence speed comparisons with existent numerical algorithm are also provided and our algorithm is fast.

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1. Introduction

The obstacle problem is a typical variational inequality of the first kind [1]. It appeared in many fields, such as elastoplastic torsion problems in lubrication theory, membranes in elastic theory, optimal control and option pricing, which means that research into numerical methods for solving the obstacle problem is of great significance.

Let Ω be a bounded domain with a Lipschitz boundary $\partial\Omega$. Given $f \in L^2(\Omega)$ and $\psi \in H^1(\Omega)$ with $\psi \leq 0$ on $\partial\Omega$. The obstacle problem, e.g., describes the equilibrium position of an elastic membrane occupying the domain Ω. The elastic membrane (1) passes through the boundary of Ω; (2) lies above an obstacle of height ψ; and (3) is subject to the action of a vertical force which is proportional to f [2].

We denote by u the vertical displacement component of the membrane. The set of admissible displacements is $K = \{u \in H_0^1(\Omega) | u \geq \psi \ \text{ a.e. in } \Omega\}$, which is a closed and

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convex set. According to the principle of energy minimization, the obstacle problem for the membrane can be stated as follows

$$\text{Find} \quad u \in K: \; E(u) = \inf_{v\in K} \left\{ \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \int_\Omega f v dx \right\}. \tag{1.1}$$

It has a unique solution u that belongs to $H_0^1(\Omega)$ [3]. In addition, if $\psi \in H^2(\Omega)$, then $u \in H^2(\Omega) \cap H_0^1(\Omega)$. It is well known that the solution of (1.1) is also characterized by the variational inequality

$$u \in K, \quad \int_\Omega \nabla u \cdot \nabla(v-u) dx \geq \int_\Omega f(v-u) dx, \qquad \forall v \in K. \tag{1.2}$$

If the solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$, then the following relations hold [2]:

$$u - \psi \geq 0, \qquad -\Delta u - f \geq 0, \qquad (u-\psi)(-\Delta u - f) = 0 \quad \text{a.e. in } \Omega. \tag{1.3}$$

Consequently, we have $\Omega = \Omega_1 \bigcup \Omega_2$ and

$$\begin{aligned} u - \psi > 0 \quad &\text{and} \quad -\Delta u - f = 0 \quad \text{in} \quad \Omega_1, \\ u - \psi = 0 \quad &\text{and} \quad -\Delta u - f > 0 \quad \text{in} \quad \Omega_2. \end{aligned}$$

Here Ω_2 denotes the coincidence set where the elastic membrane contacts with the obstacle and Ω_1 denotes the non-coincidence set.

Various algorithms have been proposed for solving the obstacle problem. In the following, we provide a brief review of some of the existing methods. The most well-known solution techniques are projected methods such as the relaxation method [4] and multigrid method [5–7], whose convergence rate depend on the mesh refinement. [8] gave active set strategies which can be efficiently implemented by the multigrid approach. [9] proposed a moving obstacle method which considered iterative approximation of the contact region. [10, 11] investigated iterative solution of piecewise linear systems for the numerical solution of the obstacle problem. [12] proposed a direct algorithm with a penalization parameter. [1] studied virtual element method which can work on very general polygonal elements, etc..

The idea of using dynamic systems to solve mathematical problems is well known. Recently, there has been increasing evidence that second-order in time dissipative systems enjoy remarkable optimization properties [13]. They have been developed for solving a variety of problems, e.g., the inverse source problem [13] and nonlinear Schrödinger equation [14]. Combined with the particle method, [15] proposed the dynamical functional particle method (DFPM) and this method is well used in solving many problems such as large linear equations [16] and eigenvalue problems [14]. It was found that this approach is very efficient. The convergence rate of DFPM is fast in all cases (exponential time) [15]. In this paper, we consider the applicability of DFPM for solving the obstacle problem.

Let us introduce the notations used in this paper firstly. Given a bounded domain $D \in R^d$ (here $d = 1, 2$) and two positive integers m and p, $W^{m,p}(D)$ is the Sobolev