

Decoupled Mixed Element Methods for Fourth Order Elliptic Optimal Control Problems with Control Constraints

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Abstract. In this paper, we study the finite element methods for distributed optimal control problems governed by the biharmonic operator. Motivated from reducing the regularity of solution space, we use the decoupled mixed element method which was used to approximate the solution of biharmonic equation to solve the fourth order optimal control problems. Two finite element schemes, i.e., Lagrange conforming element combined with full control discretization and the nonconforming Crouzeix-Raviart element combined with variational control discretization, are used to discretize the decoupled optimal control system. The corresponding a priori error estimates are derived under appropriate norms which are then verified by extensive numerical experiments.

AMS subject classifications: 49J20, 35J35, 65K10, 65N15, 65N30

Key words: Fourth order elliptic equation, optimal control problem, decoupled mixed element method, Lagrange element, nonconforming Crouzeix-Raviart element, a priori error estimates.

1. Introduction

In recent years, the optimal control problems governed by partial differential equations (PDEs) have attracted a lot of attention worldwide, and become an active research topic in related fields because of their extensive practical applications in material design, temperature control, spaceflight, hydrodynamics, aerodynamics and engineering aspects etc. Although the theory and numerical simulation for PDE-constrained control problems have been developed rapidly in the past thirty years, there is still an urgent demand in more advanced and deeper theoretical and numerical results.

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The numerical methods for optimal control problems governed by second order differential equations have been well studied, the theory and algorithms are relatively mature, see, for example, [1, 9–12, 19, 21–24, 28, 30, 37, 41, 43, 46]. We also refer to monographs [29, 35, 36, 44] for further advance. However, there are only few results on optimal control problems governed by the fourth order operator. The main difficulty of which lies in the design of proper numerical discretization method for higher order problems. For example, if the conforming element is applied directly to the fourth order elliptic equation $\Delta^2 y = f$, the finite element space must be contained in $C^1(\bar{\Omega})$, i.e., both the piecewise polynomials and their derivatives are required to be continuous across the edges of the element. However, constructing such a finite element space requires the use of fairly sophisticated finite elements. The resulting computational difficulty is undoubtedly a great challenge. Therefore, some alternative methods are proposed to alleviate the related computational difficulty, for instance the nonconforming method, in which the finite element space is only a subspace of $H^1(\Omega)$, or even a subspace of $L^2(\Omega)$ (see [13]). Generally speaking, there are two kinds of numerical methods for solving this problem: the direct method by using nonconforming finite elements (see [13, 40, 45]) and the mixed finite element method (see [5, 14, 31, 39]). Here we focus on the latter.

Concerning mixed finite element methods (FEMs) for fourth order problem, there are several well-known schemes, such as Ciarlet-Raviart method (see [14]), Herrmann-Miyoshi method (see [39]) and Hellan-Herrmann-Johnson method [26, 27, 31]. The basic idea of Ciarlet-Raviart scheme is to introduce an auxiliary variable $\omega := \Delta y$, thereby $\Delta \omega = f$. Then the fourth order problem is decomposed into a system of two second order problems. Another mixed scheme is by introducing an auxiliary variable $\phi := \nabla^2 y$ so that $\text{div div } \phi = f$ and then we obtain similar two second order problems. Both Herrmann-Miyoshi and Hellan-Herrmann-Johnson schemes rely on this form. Among these mixed schemes, Ciarlet-Raviart mixed scheme is widely used because it has a simple form. The convergence analysis of piecewise linear Ciarlet-Raviart mixed element discretization was given by Scholz [42]. Later, several mixed methods including Ciarlet-Raviart mixed element were proved to be stable with respect to some new families of mesh dependent norms by Babuška et al. [2], where error estimates are obtained in a simple and direct manner. However, all of these mixed schemes have drawbacks. In Ciarlet-Raviart mixed element scheme, although $(y, \Delta y)$ and its discrete form are stable (see [4]) in $H_0^1(\Omega) \times H^{-1}(\Delta)$ and $H_{h0}^1(\Omega) \times H_h^{-1}(\Delta_h)$, respectively, $H_h^{-1}(\Delta_h)$ is mesh-dependent. Besides, the spaces $H^{-1}(\Delta)$ and $H_h^{-1}(\Delta_h)$ are inconsistent which poses difficulty on constructing nested discrete spaces. Additionally, we may also suffer from the loss of convergence order. A similar drawback can be found for the second kind of mixed scheme (see [33]).

Recently, Zhang et al. [34, 48, 49] proposed a stable decoupled mixed element method. Based on the Helmholtz decomposition, the Hellan-Herrmann-Johnson mixed scheme (see [3]) was further decomposed such that the primal problem was translated into a coupled system of five low order equations. The stability of the discrete system is verified and the error estimation can be easily obtained. To alleviate the large scale of