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Numerical Analysis of a Dynamic Contact Problem with History-Dependent Operators

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Abstract. In this paper, we study a dynamic contact model with long memory which allows both the convex potential and nonconvex superpotentials to depend on history-dependent operators. The deformable body consists of a viscoelastic material with long memory and the process is assumed to be dynamic. The contact involves a nonmonotone Clarke subdifferential boundary condition and the friction is modeled by a version of the Coulomb's law of dry friction with the friction bound depending on the total slip. We introduce and study a fully discrete scheme of the problem, and derive error estimates for numerical solutions. Under appropriate solution regularity assumptions, an optimal order error estimate is derived for the linear finite element method. This theoretical result is illustrated numerically.

AMS subject classifications: 65M15, 65N21, 65N22
Key words: Variational-hemivariational inequality, history-dependent operators, finite element method, numerical approximation, optimal order error estimate.

1. Introduction

Variational inequalities and hemivariational inequalities play an important role in the study of various nonlinear boundary value problems arising in Mechanics, Physics, Engineering Sciences and so on. For some comprehensive references, the reader is referred to [2,9–11,13,14,17,19,22] for variational inequalities, and to [16,20,21,23,25] for hemivariational inequalities. The analysis of variational inequalities is based on monotonicity arguments and convexity theory while the analysis of hemivariational inequalities uses properties of the subdifferential in the sense of Clarke defined for locally Lipschitz functions as main ingredient and allows nonconvex functionals in formulations. Variational-hemivariational inequalities represent a special class of inequalities, where both convex and nonconvex functions are present.

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This paper is devoted to the study on numerical approximation of a general evolutional variational-hemivariational inequality involving history-dependent operators which models a dynamic contact problem with long memory. The model we consider here was first proposed in [12]. Existence and uniqueness result of the corresponding variational-hemivariational inequality are shown in [12]. In this paper, we consider numerical methods to solve the model in [12]. We derive optimal error estimates for the scheme. Since history-dependent operators appear at several places and the contact boundary conditions are of complex form, it is challenging to derive error estimates for numerical solutions of the model.

We first recall the model studied in [12]. Assume a viscoelastic body occupies a Lipschitz domain Ω in $\mathbb{R}^d$ with $d = 2,3$. We use the notation $\boldsymbol{x} = (x_i)_{i=1}^d$ for a generic point in $\overline{\Omega} = \Omega \cup \partial\Omega$ and we denote by $\boldsymbol{\nu} = (\nu_i)_{i=1}^d$ the outward unit normal on $\partial\Omega$. We denote by $\boldsymbol{u} = (u_i)$, $\boldsymbol{\sigma} = (\sigma_{ij})$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ the displacement vector, the stress tensor, and linearized strain tensor, respectively. Sometimes we do not indicate explicitly the dependence of the variables on the spatial variable $\boldsymbol{x}$. Recall that the components of the linearized strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$ are $\varepsilon_{ij}(\boldsymbol{u}) = (u_{i,j} + u_{j,i})/2$, where $u_{i,j} = \partial u_i/\partial x_j$. The indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. An index following a comma indicates a partial derivative with respect to the corresponding component of the spatial variable $\boldsymbol{x}$. A superscript prime of a variable stands for the time derivative of the variable. Moreover, we use the notation v_ν and $\boldsymbol{v}_\tau$ for the normal and tangential components of $\boldsymbol{v}$ on $\partial\Omega$ given by $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$. The normal and tangential components of the stress field $\boldsymbol{\sigma}$ on the boundary are defined by $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$, respectively. The symbol $\mathbb{S}^d$ represents the space of second order symmetric tensors on $\mathbb{R}^d$.

The boundary $\partial\Omega$ is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 and the measure of Γ_1, denoted $m(\Gamma_1)$, is positive. The body is clamped on Γ_1, so the displacement field vanishes there. Time-dependent surface tractions of density $\boldsymbol{f}_2$ act on Γ_2 and time-dependent volume forces of density $\boldsymbol{f}_0$ act in Ω. The part Γ_2 can be empty. The body is in permanent contact on Γ_3 with a device, say a piston. The contact is modeled with a nonmonotone normal damped response condition associated with a total slip-dependent version of Coulomb's law of dry friction. We are interested in the evolutionary process of the mechanical state of the body in the time interval $(0, T)$ with $T > 0$. The mathematical model of the contact problem is stated as follows.

Problem 1.1. Find a displacement field $\boldsymbol{u} : \Omega \times (0,T) \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times (0,T) \to \mathbb{S}^d$ such that for all $t \in (0,T)$,

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}'(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{C}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}'(s))\,ds \qquad \text{in } \Omega, \tag{1.1a}$$

$$\rho\boldsymbol{u}''(t) = \mathrm{Div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) \qquad \text{in } \Omega, \tag{1.1b}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{1.1c}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \qquad \text{on } \Gamma_2, \tag{1.1d}$$