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PPIFE Method with Non-Homogeneous Flux Jump Conditions and Its Efficient Numerical Solver for Elliptic Optimal Control Problems with Interfaces

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Abstract. In this paper, we design a partially penalized immersed finite element method for solving elliptic interface problems with non-homogeneous flux jump conditions. The method presented here has the same global degrees of freedom as classic immersed finite element method. The non-homogeneous flux jump conditions can be handled accurately by additional immersed finite element functions. Four numerical examples are provided to demonstrate the optimal convergence rates of the method in L^∞, L^2 and H^1 norms. Furthermore, the method is combined with post-processing technique to solve elliptic optimal control problems with interfaces. To solve the resulting large-scale system, block diagonal preconditioners are introduced. These preconditioners can lead to fast convergence of the Krylov subspace methods such as GMRES and are independent of the mesh size. Four numerical examples are presented to illustrate the efficiency of the numerical schemes and preconditioners.

AMS subject classifications: 65K15, 65N22, 65N30

Key words: Immersed finite element, interface, optimal control, preconditioners, fast solver.

1. Introduction

In recent years, the field of partial differential equation (PDE) constrained optimization has received a significant impulse due to its wide applicability, such as heat phenomena, fluid flow and numerical weather prediction. A lot of theoretical results have been developed [1]. Generally, it is difficult to obtain the analytical solutions for optimal control problems with PDEs. Actually, only approximate solutions or numerical

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solutions can be expected. Therefore, many numerical methods have been proposed to solve the elliptic optimal problems with continuous coefficients, see [2–7].

In compound materials design or fluid flow control, the elliptic equations with discontinuous coefficients are often used to model these problems. To solve the interface problems, the fitted finite element method is considered in some literatures [8, 9]. The advantage of the method is that satisfactory accuracy can be achieved. However, some efficient solvers are more efficient on the structured Cartesian mesh than on the unstructured body-fitting mesh [10]. Some physical models are more efficient on structured Cartesian meshes, such as the particle-in-cell method for plasma simulation due to the "scattering" and "gathering" steps in this method [11–13]. Also, for moving interface problems, it takes additional cost to reform the mesh at each time step in order to fit the moving interface [14–16].

Since Peskin's pioneering work of the immersed boundary method [17], a large number of numerical methods have been constructed based on finite difference discretization [18,19]. The immersed boundary method is simple and robust and has been applied to many problems in mathematical biology and computational fluid mechanics [20–22]. But the computed solution is only first-order accuracy and is smeared in the neighborhood of the interface. In [23, 24], non-traditional finite element methods for solving elliptic and parabolic equations with interfaces are proposed. It achieved second-order accuracy in the L^∞ norm. However, the resulting linear system of equations is not symmetric due to the different trial and test functions. Also, there has been a large body of work from other unfitted numerical methods for solving the interface problems [25–29].

The immersed finite element (IFE) methods are a particular class of finite element methods based on Cartesian meshes. The basic idea of IFE methods is to employ standard finite element functions on non-interface elements and modify the approximating functions on interface elements. The IFE methods have been used for solving second-order elliptic interface problems [30, 31], the planar elasticity systems [32] and the parabolic interface problems [15]. Numerical experiments have demonstrated that the classic IFE methods often have a much larger pointwise error over interface elements. To overcome the difficulties, the partially penalized IFE (PPIFE) methods are proposed in some references [33–36]. The methods can improve the classic IFE methods and obtain smaller pointwise error. But the methods are developed for solving interface problems with the homogeneous flux jump conditions.

The elliptic interface problems with non-homogeneous flux jump conditions can model chemical reaction diffusion processes in which the reaction takes place only on interface due to the effects of catalyst. This leads to the chemical concentration to be continuous but the gradient of the concentration to have a jump on the interface. In [37], the homogenization technique based on the level set idea is proposed to deal with the non-homogeneous flux jump conditions. In [38, 39], the new piecewise polynomials are constructed to approximate the non-homogeneous flux jump conditions. However, the ultimate goal is not only the numerical simulation of the chemical reaction diffusion processes, but rather the optimal control of the considered process.