Numer. Math. Theor. Meth. Appl. Vol. **13**, No. 3, pp. 788-813
doi: 10.4208/nmtma.OA-2019-0121 August 2020

Energy Identities and Stability Analysis of the Yee Scheme for 3D Maxwell Equations

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Received 13 August 2019; Accepted (in revised version) 11 December 2019

Abstract. In this paper numerical energy identities of the Yee scheme on uniform grids for three dimensional Maxwell equations with periodic boundary conditions are proposed and expressed in terms of the L^2, H^1 and H^2 norms. The relations between the H^1 or H^2 semi-norms and the magnitudes of the curls or the second curls of the fields in the Yee scheme are derived. By the L^2 form of the identity it is shown that the solution fields of the Yee scheme is approximately energy conserved. By the H^1 or H^2 semi norm of the identities, it is proved that the curls or the second curls of the solution of the Yee scheme are approximately magnitude (or energy)-conserved. From these numerical energy identities, the Courant-Friedrichs-Lewy (CFL) stability condition is re-derived, and the stability of the Yee scheme in the L^2, H^1 and H^2 norms is then proved. Numerical experiments to compute the numerical energies and convergence orders in the L^2, H^1 and H^2 norms are carried out and the computational results confirm the analysis of the Yee scheme on energy conservation and stability analysis.

AMS subject classifications: 65M06, 65M12, 65N06, 65N12

Key words: Finite difference time domain (FDTD) method, Maxwell equations, energy conservation, stability, convergence, CFL.

1. Introduction

Yee scheme, introduced by Yee [1] in 1966, is a very popular and efficient finite difference method for solving time dependent Maxwell equations, and has been applied in a broad range of problems in industry by combining absorbing boundary condition [2-6] etc. This scheme is based on the fully staggered grids in space and time, second order accurate and with many merits [7-11]. Over fifty years Yee scheme attracted many scientists' interests producing many good research work [12-20] and their references therein). For example, Taflove and Brodwin [12] proved by

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the Fourier method that Yee scheme is stable if the Courant-Friedrichs-Lewy (CFL) condition, $c\Delta t < 1/\sqrt{(1/\Delta x)^2 + (1/\Delta y)^2 + (1/\Delta z)^2}$ (in 3D case, where $c = 1/\sqrt{\varepsilon\mu}$ is the wave velocity), is satisfied. Monk and Süli [13] proved that Yee scheme is stable and second-order convergent in the L^2 norm, and showed in [14] by numerical experiments that Yee scheme is super convergent. Liu [15] verified that the staggered grids in the Yee scheme make numerical dispersion error smaller than the non-staggered ones. Remis [16] studied practical stability conditions of FDTD on nonuniform tensor product grids. Fang and Ying [17] analyzed stability of the FDTD scheme to UPML for time dependent Maxwell equations. The second order convergence and stability analysis of the Yee scheme on uniform grids in the H^1 norm were given in [18]. Recently, Li and Shields [19] proved that the Yee scheme for the Maxwell equations in metamaterials is of second order super convergence in the L^2 norm, and Londersele etc. [20] studied the stability of nonuniform FDTD combined with novel local implicitization techniques.

In order to overcome the CFL restriction on time and spatial step sizes and improve accuracy of the Yee scheme, many methods [21-27] etc. were proposed and analyzed rigorously. For example, the ADI-FDTD method proposed by Zheng etc. [21] and Namiki [22], the energy conserved splitting FDTD (EC-S-FDTD) methods by Chen etc. [23, 24], the splitting multi-symplectic method by Kong etc. [25], the energy-dissipation splitting method by Hong etc. [26] and the high-accuracy energy-preserving S-AVF methods proposed by Cai [27] etc., all these methods were proved to be unconditionally stable or overcome the CFL condition for the Yee scheme. Apart from the improvement of stability, there are many methods which improve the accuracy of the Yee scheme [28-35] and the references therein.

Energy conservation is an important property of electromagnetic fields. Study of energy conservation of numerical methods of Maxwell equations is an important research topic and causes many researchers' interests [23-27, 30-40] and the references therein. For example, L^2 forms of the energy identities for the EC-S-FDTD scheme in [23, 24], the splitting multi-symplectic scheme in [25], the S-AVF methods [27], and the spatial fourth order energy-conserved S-FDTD scheme [30] etc. were proposed and stability of these schemes were then proved. It was also proved in [36] that there exists the H^1 semi-form of identity for the EC-S-FDTD method. Similar research work about the numerical energy identities (containing perturbation terms) of the ADI-FDTD method was given in [37]. Recently, the energy identities of the 2D EC-S-FDTD method with periodic boundary conditions in terms of the H^2 norm were proposed in [39] and interpretation of these identities in H^1 and H^2 semi-norms was given. However, energy identity of the Yee scheme in L^2 or H^k $(k = 1, 2)$ is not available in the referenced papers.

In this paper, we focus on energy conservation analysis of the Yee scheme for the 3D Maxwell equations with periodic boundary conditions. Numerical energy identities in terms of the L^2, H^1 and H^2 norms for this scheme are derived by the energy methods, and interpretation of these identities is provided by studying the H^1 and H^2 semi-norms of the fields. By these identities, we prove that the Yee scheme is approximately energy and curl magnitude preserving since the numerical energy identities contain