

An Adaptive Hybrid Spectral Method for Stochastic Helmholtz Problems

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Abstract. The implementation of an adaptive hybrid spectral method for Helmholtz equations with random parameters is addressed in this work. New error indicators for generalized polynomial chaos for stochastic approximations and spectral element methods for physical approximations are developed, and systematic adaptive strategies are proposed associated with these error indicators. Numerical results show that these error indicators provide effective estimates for the approximation errors, and the overall adaptive procedure results in efficient approximation method for the stochastic Helmholtz equations.

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1. Introduction

During the last decade there has been a rapid development in efficient stochastic Galerkin approximation methods for solving partial differential equations (PDEs) with random parameters. The efficiency of Galerkin methods relies on properly choosing basis functions of finite dimensional projection spaces. There are two projection spaces involved herein: the stochastic approximation space and the physical approximation space. In the literature, a popular choice for the stochastic approximation is generalized polynomial chaos (gPC) [18, 37, 39], which is a stochastic spectral method.

Typical choices for the physical approximation are finite element methods [4, 13], spectral methods [8, 32] and spectral element methods [30]. When the solution is sufficiently regular, spectral methods have exponential convergence rates. However, if the solution is not smooth, the rates of spectral methods deteriorate. To result in

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an efficient approximation in general case, adaptive versions of spectral methods are actively developed.

To resolve discontinuities of solutions with respect to the random parameters, the multi-element generalized polynomial chaos method is first developed [36]. After that, a model reduction based mesh refinement method for stochastic approximations is proposed [28, 29]. For an overall adaptive procedure, adaptive stochastic Galerkin finite element methods are developed in [10], where the gPC degree and dimension adaption and the finite element mesh refinement for the physical domain are conducted based on a residual-based a posteriori error estimator. For efficient adaptive procedures, effective local problem based error estimators for stochastic Galerkin finite elements are developed in [2, 3].

As spectral methods are expected to have higher order accuracy than low order finite element methods, in this work, we develop a new adaptive hybrid spectral method as an alternative to the adaptive stochastic Galerkin finite elements. In this hybrid spectral method, the stochastic domain is discretized by gPC and the physical domain is discretized by the spectral element method, which gives flexibility to conduct local refinements. Since gPC and the (physical) spectral element are both spectral methods, they are both expected to have high order convergence rates. The novelty of this work lies on new effective error indicators and adaptive strategies for both gPC degree adaption and spectral element refinement.

To illustrate the framework and the efficiency of our new approach, the stochastic Helmholtz equations are studied as a benchmark problem, which plays an important role in ocean acoustics, optics and electromagnetics [21, 24, 42]. The random sources of Helmholtz problems typically arise from lack of knowledge or measurement of material refractive indices or wave number parameters. This paper is organized as follows. In Section 2, the detailed setting of stochastic Helmholtz equations is introduced, and the frameworks of stochastic Galerkin methods and gPC are discussed. We also discuss the spectral element methods for physical approximations in Section 2. Our main adaptive hybrid spectral methods for both stochastic and physical approximations are presented in Section 3. Numerical results are discussed in Section 4. Section 5 concludes the paper.

2. Problem setting and generalized polynomial chaos (gPC)

This section describes the mathematical setting of stochastic Helmholtz equations, its variational formulation, and gPC approximations in the stochastic space.

2.1. Problem setting

Let $D \subset \mathbb{R}^2$ be a physical domain which is open, bounded, connected and with a polygonal boundary ∂D , and $\boldsymbol{x} = [x_1, x_2]^T \in \mathbb{R}^2$ denote a physical variable. Let $\boldsymbol{\xi} = [\xi_1, \dots, \xi_n]^T$ be a random vector, in which the random variables ξ_1, \dots, ξ_n are